Partial Differential Equations (TATA27)
Spring Semester 2015
Homework 7

7.1 (a) Suppose that $\phi\colon \mathbf{R} \to \mathbf{R}$ is a bounded odd function. Show that if u is given by (7.3) (so is a solution to (7.2)) then $u(\cdot, t)$ is also odd for each $t > 0$.

(b) Now suppose that ϕ is bounded and even. Prove that $u(\cdot, t)$ given by (7.3) is also even for each $t > 0$.

7.2 (a) Use the ideas of reflections from Section 6.4.1 and 7.1(a) to solve the following boundary and initial value problem on the half line:

$$\begin{cases} \partial_t u(x,t) - \partial_{xx} u(x,t) = 0 & \text{for } x \in (0,\infty) \text{ and } t > 0, \\ u(x,0) = \phi(x) & \text{for } x \in (0,\infty), \text{ and} \\ u(0,t) = 0 & \text{for } t > 0. \end{cases} \tag{1}$$

(b) Further develop these ideas, just as we did in Section 6.4.2, to solve (7.4) via an alternative method to the separation of variables we used in Section 7.5.

(c) Make use of 7.1(b) to help you solve a similar problem to (1):

$$\begin{cases} \partial_t u(x,t) - \partial_{xx} u(x,t) = 0 & \text{for } x \in (0,\infty) \text{ and } t > 0, \\ u(x,0) = \phi(x) & \text{for } x \in (0,\infty), \text{ and} \\ \partial_x u(0,t) = 0 & \text{for } t > 0. \end{cases}$$

Here we replaced the Dirichlet boundary condition $u(0,t) = 0$ with $\partial_x u(0,t) = 0$, which is called a *Neumann condition*.

7.3 In Section 8.1 we estimated the error between derivatives and finite differences in terms of the mesh size δx for a $C^4(\mathbf{R})$ function.

(a) If u is merely a $C^3(\mathbf{R})$ function, what is the error between its first derivative and its centred difference?

(b) If u is merely a $C^2(\mathbf{R})$ function, what is the error between its first derivative and its centred difference?

(c) If u is merely a $C^3(\mathbf{R})$ function, what is the error between its second derivative and its centred second difference?

7.4 Suppose $u \in C^5(\mathbf{R})$. Can you approximate the first derivative $u'(x)$ using a similar method we used in lectures with an error of $O((\delta x)^4)$? [Hint: Make use of the function u evaluated at the points $x + k(\delta x)$ for $k = -2, -1, 1, 2$.]