Partial Differential Equations (TATA27)
Spring Semester 2017
Homework 3

3.1 Consider two points $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2$ with polar coordinates (r, θ) and (a, ϕ), respectively. Using a geometric argument (or otherwise) show that

$$|\mathbf{x} - \mathbf{y}|^2 = r^2 - 2ar\cos(\theta - \phi) + a^2.$$

Use this fact to help you rewrite the Poisson formula

$$u(r,\theta) = \frac{(a^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2} d\phi \tag{5.6}$$

as

$$u(\mathbf{x}) = \frac{(a^2 - |\mathbf{x}|^2)}{2\pi a} \int_{|\mathbf{y}|=a} \frac{\tilde{h}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} d\sigma(\mathbf{y}). \tag{5.7}$$

3.2 Let $W = \{\mathbf{x} = (r,\theta) \in \mathbf{R}^2 \,|\, 0 < r < a \text{ and } 0 < \theta < \beta\}$ denote a wedge of length a and angle β (where (r, θ) are polar coordinates). Using the same procedure as we used to derive the Poisson formula for D derive a analogous formula for the solution u to

$$\begin{cases} \Delta u = 0 & \text{in } W, \\ u(r,0) = u(r,\beta) = 0 & \text{for } r \in (0,a), \text{ and} \\ u(a,\theta) = h(\theta) & \text{for } \theta \in (0,\beta), \end{cases}$$

3.3 Green's second identity says that for two functions $u, v \in C^2(\overline{\Omega})$

$$\int_\Omega u(\mathbf{x})\Delta v(\mathbf{x}) - v(\mathbf{x})\Delta u(\mathbf{x}) d\mathbf{x} = \int_{\partial\Omega} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) - v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) d\sigma(\mathbf{x}). \tag{5.10}$$

Use Green's first identity (5.8) to prove (5.10).

3.4 Consider the function $\Phi\colon \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ defined by

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\alpha(2)} \ln|\mathbf{x}| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\mathbf{x}|^{n-2}} & \text{if } n > 2, \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbf{R}^n$. (So, in particular, $\alpha(2) = \pi$ and $\alpha(3) = 4\pi/3$.)

(a) Prove that Φ is harmonic on $\mathbf{R}^n \setminus \{\mathbf{0}\}$.

(b) Consider the domain $B_r(\mathbf{0}) = \{\mathbf{y} \in \mathbf{R}^n \,|\, |\mathbf{y}| < r\}$. Then the outward unit normal at $\mathbf{x} \in \partial B_r(\mathbf{0})$ is $\mathbf{n}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$. Prove that

$$\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{x}) = \frac{-1}{n\alpha(n)} \frac{1}{|\mathbf{x}|^{n-1}}$$

for each $n = 1, 2, \ldots$.

3.5 Let Ω be an open set with C^1 boundary, and let $f\colon \Omega \to \mathbf{R}$ and $h\colon \partial\Omega \to \mathbf{R}$. Use Green's first identity (5.8) to prove the uniqueness of solutions $u \in C^2(\overline{\Omega})$ to the following boundary value problems.

(a)

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega \end{cases}$$

(b)

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} + au = h & \text{on } \partial\Omega \end{cases}$$

where $\partial u/\partial \mathbf{n} := \mathbf{n} \cdot \nabla u$, $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ and $a > 0$ is a constant.

3.6 Consider a solution $u \in C^2(\overline{\Omega})$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} = h & \text{on } \partial\Omega \end{cases}$$

Observe that for any $c \in \mathbf{R}$ $u + c$ is also a solution. Could there be any more $C^2(\overline{\Omega})$ solutions?