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Tentamen TATA 27 Partial Differential Equations
22 Oktober 2007, 14-19

You can use on this examination Formelsamlingen i partiella diff.ekv. för M3 av L-E Andersson and dictionaries. No calculators.

1. Consider the following Dirichlet problem for the heat equation on the half-line:
$$u_t - ku_{xx} = 0 \quad \text{for } t > 0 \text{ and } 0 < x < \infty$$
and
$$u(0,t) = 0 \quad \text{for } t > 0,$$
supplied with the initial condition
$$u(x,0) = \begin{cases} x^2 & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1. \end{cases}$$
Calculate $u_x(0,t)$ and show that this function is strictly decreasing and tends to 0 as t tends to ∞.

2. Solve the following boundary value problem for the wave equation:
$$u_{tt} = u_{xx} \quad \text{for } 0 < x < \pi \text{ and } t > 0,$$
$$u(0,t) = u_x(\pi,t) = 0 \quad \text{for } t > 0$$
and
$$u(x,0) = 1 \quad \text{and} \quad u_t(x,0) = \sin(x/2) \quad \text{for } 0 < x < \pi.$$

3. Consider the function $u = u(x,y)$ which satisfies the Poisson equation
$$\Delta u = 1 \quad \text{for } x^2 + y^2 < 1$$
and the following Dirichlet condition
$$u(x,y) = \cos^2(x+y) \quad \text{for } x^2 + y^2 = 1.$$
Show that
$$-1/2 \le u(x,y) - (x^2+y^2+1)/4 \le 1/2 \quad \text{for } x^2 + y^2 \le 1.$$

4. Find
$$\min \int_2^4 (u^2 + u\,u' + 2u'^2)dx,$$
where minimum is taken over all functions satisfying $u(2) = 2$ and $u(4) = 3$.

5. Solve the problem

$$-u''(x) + 4u(x) = \delta(x-3) \quad \text{for } x > 0, \quad u(0) = 0,$$

and $u(x) \to 0$ as $x \to \infty$. (Hint: use Fourier transform to find a particular solution to the equation)

6. Let $D = \{(x, y, z) \in \mathbb{R}^3 : 1 < x^2 + y^2 + z^2 < 5\}$. Prove that the heat equation

$$u_t - \Delta u = x \quad \text{in } D \text{ for } t > 0,$$

with boundary conditions

$$u = 1 \text{ for } x^2 + y^2 + z^2 = 1, \quad \frac{\partial u}{\partial \hat{n}} = 0 \text{ for } x^2 + y^2 + z^2 = 5 \text{ and for } t > 0$$

and with the initial condition

$$u = 0 \quad \text{for } t = 0$$

has at most one solution. Here $\hat{n}$ is the unit outward normal to the boundary of D. (Hint:use the energy method)