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Kurskod: TATA27
Provkod: TEN1

Tentamen i Partiella differentialekvationer TATA27

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You are allowed to use the formula sheet for TATA27 which is handed out at the exam. No calculators allowed.

1. Solve
$$\begin{cases} u_t(x,t) = 3u_{xx}(x,t), & 0 < x < 2, t > 0, \\ u_x(0,t) = 0 = u_x(2,t), & t > 0, \\ u(x,0) = x^2, & 0 < x < 2, \end{cases}$$
for $u(x,t)$, $0 < x < 2$, $t > 0$. Calculate $\lim_{t\to\infty} u(x,t)$ for $0 < x < 2$.

2. (a) Show that any solution $u(x,t)$ to the one dimensional wave equation $u_{tt} = c^2 u_{xx}$ is of the form
$$u(x,t) = f(x - ct) + g(x + ct),$$
for some one-variable functions $f(s)$ and $g(s)$.

   (b) Use the result from (a) to solve the initial/boundary value problem
$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & x > 0, t > 0, \\ u(0,t) = b(t), & t > 0, \\ u(x,0) = a(x), & x > 0, \\ u_t(x,0) = 0, & x > 0, \end{cases}$$
for given functions $a(s)$, $b(s)$, $s > 0$.

3. (a) Find the Green's function
$$G_D\big((x,y),(a,b)\big)$$
for the first quadrant $D := \{(x,y) \ ; \ x, y > 0\}$ with pole at $(a,b) \in D$.
Hint: The points (a,b), $(-a,b)$, $(a,-b)$ and $(-a,-b)$ are relevant.

   (b) Solve the Dirichlet problem
$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0, & x > 0, y > 0, \\ u(x,0) = h(x), & x > 0, \\ u(0,y) = 0, & y > 0, \end{cases}$$
for any given function $h(x)$, $x > 0$.

4. Find approximations to the first two Dirichlet eigenvalues for the Laplace operator on the unit disk $D := \{(x,y) \ ; \ x^2 + y^2 < 1\}$, by using Rayleigh–Ritz's method with the two test functions $f_1(x,y) = 1 - \sqrt{x^2+y^2}$ and $f_2(x,y) = x(1 - \sqrt{x^2+y^2})$.
Hint: In polar coordinates the gradient is $\nabla u = u_r \hat{r} + r^{-1} u_\varphi \hat{\varphi}$.

5. (a) Assume that $u(x,y)$ is a solution to the partial differential equation

$$u_{yy} = -u_{xx}$$

on the square $D := \{(x,y) \ ; \ 0 < x < 1, 0 < y < 1\}$ and that $u(1/2, 1/2) = 1$. Is there necessarily a point (x_0, y_0) on the boundary ∂D where $u(x_0, y_0) \geq 0$? If so, determine the smallest subset $E \subset \partial D$ which necessarily contains such a point (x_0, y_0).

(b) Same problem for the equation $u_{yy} - u_{xx} = 0$.

(c) Same problem for the equation $u_{yy} + u_x = 0$.

Motivate your answers. You only need to give full proofs of your statements for one of the three equations. (Which one is your choice.)

6. Assume that u is a twice continuously differentiable function in a domain $D \subset \mathbf{R}^2$ with satisfies the mean value property, that is

$$f(r) = \frac{1}{2\pi} \int_0^{2\pi} u(x + r\cos\varphi, y + r\sin\varphi) d\varphi$$

equals $u(x,y)$ for any r less than the distance from (x,y) to ∂D, for any $(x,y) \in D$. Prove that u is a harmonic function in D.