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Svar till Tentamen TATA 27/TEN1 Partial Differential Equations
28 May 2011, 08-13

1. We are looking for u in the form $u = 1 + v$. Then v satisfies

$$v_{tt} = 4v_{xx} \text{ for } x > 0 \text{ and } t > 0,$$
$$v(0,t) = 0 \text{ for } t > 0$$

and

$$v(x,0) = x - 1 \text{ and } v_t(x,0) = x^2 \text{ for } x > 0.$$

Now we can use the odd extension and d'Alembert's formula, as the result we obtain

$$v(x,t) = x - 1 + \frac{1}{12}\big((x+2t)^3 - (x-2t)^3\big) \text{ for } x > 2\,t$$

and

$$v(x,t) = x + \frac{1}{12}\big((x+2t)^3 - (2t-x)^3\big) \text{ for } x < 2\,t.$$

Therefore, $u(1,1) = 2 + \frac{1}{12}(3^3 - 1) = \frac{25}{6}$.

2. We are looking for the solution u in the form $u = 1 - x + v$. Then v satisfies

$$v_t = 4v_{xx} \quad \text{for } 0 < x < 2 \text{ and } t > 0,$$
$$v(0,t) = 0 \text{ and } v_x(2,t) = 0 \text{ for } t > 0$$

and

$$v(x,0) = x - 1 \text{ for } 0 < x < 2.$$

Now, we can use the method of separation of variables, which gives

$$v(x,y) = \sum_{k=0}^{\infty} A_k e^{-4\lambda_k t} \sin \sqrt{\lambda_k} x,$$

where

$$\lambda_k = \pi/4 + k\pi/2, k = 0,1,2,\ldots \quad \text{and} \quad A_k = \int_0^2 (x-1) \sin \sqrt{\lambda_k} x dx.$$

3. We cannot apply the maximum principle for the non-homogeneous Poisson equation. We represent u as $u = x^2 + y^2 + v$. Then v satisfies

$$v_{xx} + v_{yy} = 0 \text{ for } 0 < x < 1 \text{ and } 0 < y < 2$$

and is subject to the Dirichlet boundary conditions

$$u(x,0) = x - x^2 \text{ for } 0 < x < 1, \; u(1,y) = 2y - y^2 \text{ for } 0 < y < 2,$$
$$u(x,2) = 3x - 2 - x^2 \text{ for } 0 < x < 1 \text{ and } u(0,y) = y - y^2 \text{ for } 0 < y < 2.$$

Applying the maximum principle we get $-2 \le v(x,y) \le 1$ which implies

$$-2 + x^2 + y^2 \le u(x,y) \le 1 + x^2 + y^2.$$

4. Equation for u is $u'' - u = e^x/2$. Its general solution is

$$u(x) = \frac{xe^x}{4} + ae^x + be^{-x}.$$

Using the first boundary condition, we obtain

$$u(x) = \frac{(x-1)e^x}{4} + a(e^{x-1} - e^{-x+1}).$$

Taking into account the second boundary condition we get

$$a = \frac{4(e^{-3} - e^3)}{3e^4}$$

5. The answer is

$$f'' = g(x) + 2\delta(x-2) + 2\delta(x-3),$$

where $g(x) = 2$ if $x < 2$ or $x > 3$ and $g(x) = -2$ if $2 < x < 3$.

6. Assume that there are two solutions u_1 and u_2 of the problem. Then $u = u_2 - u_1$ satisfies

$$u_{tt} - u_{xx} - u_{yy} = 0 \quad \text{for } x^2 + y^2 < 2\ ,\ t > 0, \tag{1}$$
$$u(x, y, t) = 0 \quad \text{for } x^2 + y^2 = 2,\ t > 0$$

and

$$u(x, y, 0) = u_t(x, y, 0) = 0 \quad \text{for } x^2 + y^2 < 2.$$

Multiplying (1) by u_t and integrating over the disc $D = \{(x, y) \,:\, x^2 + y^2 < 2\}$ we get

$$\int_D (u_{tt} - u_{xx} - u_{yy}) u_t dxdy = 0. \tag{2}$$

Using that

$$\int_D u_{tt} u_t dxdy = \frac{1}{2}\frac{d}{dt}\int_D u_t^2 dxdy$$

and

$$-\int_D (u_{xx} + u_{yy})\ u_t dxdy = \int_D (u_x\ u_{xt} + u_y\ u_{yt}) dxdy = \frac{1}{2}\frac{d}{dt}\int_D (u_x^2 + u_y^2) dxdy,$$

we derive from (2) that

$$\frac{d}{dt}\Big(\int_D (u_t^2 + u_x^2 + u_y^2) dxdy\Big) = 0,$$

which implies

$$\int_D (u_t^2 + u_x^2 + u_y^2) dxdy = c \quad \text{for all } t,$$

where c is a constant. Using zero initial conditions for u we conclude that the left-hand side of the last equality is zero for $t = 0$ and hence the constant c is also zero. This implies that $u_t = 0$ and $u_x = u_y = 0$ and therefore u is a constant. Since $u = 0$ on the boundary we obtain $u = 0$ everywhere.