Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions to Homework 9

9.1 (a) See Figure 1.

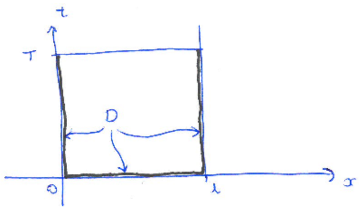

Figure 1: Here is a picture of the set D.

(b) The motivation behind why the wave equation does not satisfy a maximum principle is that wave profiles travelling in opposite directions may collide with one another and thus be larger in value that the initial data — in physics this is called *constructive interference*. We want to take this idea and apply it to construct a specific example.

We want our solution to be two waves which travel towards each other. Let $\phi \in C^\infty(\mathbf{R})$ be a function such that $|\phi(x)| \leq 1$ and

$$\phi(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0) \cup (\ell/2, \infty), \\ 1 & \text{for } x \in (\ell/6, 2\ell/6). \end{cases}$$

Then set $\phi_+(x) = \phi(x)$ and $\phi_-(x) = \phi(x - \ell/2)$. Clearly $u(x,t) = \phi_+(x-t) + \phi_-(x+t)$ solves the heat equation, so to check whether or not it solves (1) we must check the initial and boundary conditions. We have $u(x,0) = \phi_+(x) + \phi_-(x)$ and $\partial_t u(x,t) = -\phi_+'(x) + \phi_-'(x)$, so we choose

$$g(x) = \phi_+(x) + \phi_-(x) \quad \text{and} \quad h(x) = -\phi_+'(x) + \phi_-'(x). \tag{‡}$$

We also have $u(0,t) = \phi_+(-t) + \phi_-(t)$, which is zero provided $t - \ell/2 \leq 0 \iff t \leq \ell/2$. Finally, $u(\ell,t) = \phi_+(\ell - t) + \phi_-(\ell + t)$, which is zero provided $\ell - t \geq \ell/2 \iff t \leq \ell/2$. Therefore, u solves (1) with g and h as in (‡) and $T = \ell/2$.

9.2 (a) **Theorem.** Suppose $\Omega \subset \mathbf{R}^n$ is an open bounded connected set and $T > 0$. Let $u\colon \overline{\Omega} \times [0,T] \to \mathbf{R}$ be a continuous function which is also a solution to the heat equation $\partial_t u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = 0$ for $(\mathbf{x},t) \in \Omega \times (0,T]$. Then the minimum value of u is attained at a point $(\mathbf{x},t) \in \overline{\Omega} \times [0,T]$ such that either $t = 0$ or $\mathbf{x} \in \partial\Omega$.

Proof. Observe that if u satisfies the hypothesis of Theorem 7.1, then so does $-u$. Since the maximum value of $-u$ is the minimum value of u we can apply Theorem 7.1 to $-u$ and conclude that u attains its minimum value at a point $(\mathbf{x},t) \in \overline{\Omega} \times [0,T]$ such that either $t = 0$ or $\mathbf{x} \in \partial\Omega$.

(b) Suppose we have two solutions u_1 and u_2 to (7.1). Then $v = u_2 - u_1$ solves

$$\begin{cases} \partial_t v(\mathbf{x},t) - \Delta v(\mathbf{x},t) = 0 & \text{for } \mathbf{x} \in \Omega \text{ and } t \in (0,T]; \\ u(\mathbf{x},0) = 0 & \text{for } \mathbf{x} \in \overline{\Omega}; \text{ and} \\ u(\mathbf{y},t) = 0 & \text{for } \mathbf{y} \in \partial\Omega \text{ and } t \in (0,T]. \end{cases}$$

Theorem 7.1 and 9.2(a) say that

$$\max_{(\mathbf{x},t) \in \overline{\Omega} \times [0,T]} |v(x,t)| = \max_D |v(x,t)|$$

where $D = (\overline{\Omega} \times \{0\}) \cup (\overline{\Omega} \times (0,T])$. But clearly $\max_D |v(x,t)| = 0$, so $v \equiv 0$ and so $u_1 \equiv u_2$.

(c) Now the difference $u_2 - u_1 = v$ solves

$$\begin{cases} \partial_t v(\mathbf{x},t) - \Delta v(\mathbf{x},t) = 0 & \text{for } \mathbf{x} \in \Omega \text{ and } t \in (0,T]; \\ u(\mathbf{x},0) = \phi_2(\mathbf{x}) - \phi_1(\mathbf{x}) & \text{for } \mathbf{x} \in \overline{\Omega}; \text{ and} \\ u(\mathbf{y},t) = g_2(\mathbf{y},t) - g_1(\mathbf{y},t) & \text{for } \mathbf{y} \in \partial\Omega \text{ and } t \in (0,T]. \end{cases}$$

and again Theorem 7.1 and 9.2(a) say

$$\max_{\mathbf{x}\in\overline{\Omega}, t\in[0,T]} |u_2(\mathbf{x},t) - u_1(\mathbf{x},t)| = \max_{(\mathbf{x},t)\in\overline{\Omega}\times[0,T]} |v(x,t)| = \max_D |v(x,t)|$$

But

$$\begin{aligned} \max_D |v(x,t)| &\leq \max_{\mathbf{x}\in\overline{\Omega}} |v(\mathbf{x},t)| + \sup_{\mathbf{x}\in\partial\Omega, t\in(0,T]} |v(\mathbf{x},t)| \\ &= \max_{\mathbf{x}\in\overline{\Omega}} |\phi_2(\mathbf{x}) - \phi_1(\mathbf{x})| + \sup_{\mathbf{x}\in\partial\Omega, t\in(0,T]} |g_2(\mathbf{x},t) - g_1(\mathbf{x},t)|. \end{aligned}$$

Use Theorem 7.1 and 9.2(a) to prove the following stability result: If u_1 and u_2 both solve (7.1) with the initial conditions ϕ_1 and ϕ_2 and boundary conditions g_1 and g_2, respectively, then

$$\max_{\mathbf{x}\in\overline{\Omega}, t\in[0,T]} |u_2(\mathbf{x},t) - u_1(\mathbf{x},t)| \leq \max_{\mathbf{x}\in\overline{\Omega}} |\phi_2(\mathbf{x}) - \phi_1(\mathbf{x})| + \sup_{\mathbf{x}\in\partial\Omega, t\in(0,T]} |g_2(\mathbf{x},t) - g_1(\mathbf{x},t)|.$$

Putting these estimates together gives us the required stability result.

9.3 We have

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} \frac{1}{2\sqrt{\pi t}} e^{-y^2/4t} dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{4\pi t} e^{-(x^2+y^2)/4t} dxdy \\ &= \int_0^{2\pi}\int_0^{\infty} \frac{r}{4\pi t} e^{-r^2/4t} dr d\theta \\ &= \int_0^{\infty} \frac{r}{2t} e^{-r^2/4t} dr = -e^{-r^2/4t}\Big|_0^{\infty} = 1. \end{aligned}$$

Therefore $I = 1$ and, since $x \mapsto S(x,t)$ is even $\int_0^\infty S(x,t)dx = 1/2$.

9.4 (a) Recall the definitions from Section 1.2. We need to check that the operator $\partial_t - \Delta$ is a linear operator. Take two functions u and v and two constants α and β. Then,

$$\begin{aligned} (\partial_t - \Delta)(\alpha u + \beta v) &= (\partial_t - \sum_{j=1}^n \partial_{x_j x_j})(\alpha u + \beta v) \\ &= \partial_t(\alpha u + \beta v) - (\sum_{j=1}^n \partial_{x_j x_j})(\alpha u + \beta v) \\ &= \partial_t(\alpha u + \beta v) - \sum_{j=1}^n \partial_{x_j x_j}(\alpha u + \beta v) \\ &= (\alpha\partial_t u + \beta\partial_t v) - \sum_{j=1}^n \partial_{x_j}(\alpha\partial_{x_j} u + \beta\partial_{x_j} v) \\ &= (\alpha\partial_t u + \beta\partial_t v) - \sum_{j=1}^n (\alpha\partial_{x_j x_j} u + \beta\partial_{x_j x_j} v) \\ &= \alpha\left(\partial_t u - \sum_{j=1}^n \partial_{x_j x_j} u\right) + \beta\left(\partial_t v - \sum_{j=1}^n \partial_{x_j x_j} v\right) \end{aligned}$$

Thus $\mathcal{L}(\alpha u + \beta v) = \alpha\mathcal{L}u + \beta\mathcal{L}v$ where $\mathcal{L} = (\alpha u + \beta v)$, so the operator $\mathcal{L}$ is linear and hence the heat equation is linear.

(b) We know that $\partial_2 u - \partial_{11} u = 0$. By the chain rule

$$\begin{aligned}
\partial_t u(\sqrt{\alpha}x, \alpha t) &= \alpha(\partial_2 u)(\sqrt{\alpha}x, \alpha t), \\
\partial_x u(\sqrt{\alpha}x, \alpha t) &= \sqrt{\alpha}(\partial_1 u)(\sqrt{\alpha}x, \alpha t) \quad \text{and} \\
\partial_{xx} u(\sqrt{\alpha}x, \alpha t) &= \alpha(\partial_{11} u)(\sqrt{\alpha}x, \alpha t).
\end{aligned}$$

Thus

$$\begin{aligned}
\partial_t u(\sqrt{\alpha}x, \alpha t) - \partial_{xx} u(\sqrt{\alpha}x, \alpha t) &= \alpha(\partial_2 u)(\sqrt{\alpha}x, \alpha t) - \alpha(\partial_{11} u)(\sqrt{\alpha}x, \alpha t) \\
&= \alpha((\partial_2 u)(\sqrt{\alpha}x, \alpha t) - (\partial_{11} u)(\sqrt{\alpha}x, \alpha t)) \\
&= \alpha((\partial_2 u) - (\partial_{11} u))(\sqrt{\alpha}x, \alpha t) = 0.
\end{aligned}$$

9.5 (a) By applying the chain rule, we see that

$$\begin{aligned}
\partial_t u(x,t) &= -\frac{x}{4t^{3/2}} g'(x/(2\sqrt{t})) \\
\partial_x u(x,t) &= \frac{1}{2t^{1/2}} g'(x/(2\sqrt{t})) \quad \text{and} \\
\partial_{xx} u(x,t) &= \frac{1}{4t} g''(x/(2\sqrt{t})).
\end{aligned}$$

Therefore,

$$0 = \partial_t u(x,t) - \partial_t u(x,t) = -\frac{x}{4t^{3/2}} g'(x/(2\sqrt{t})) - \frac{1}{4t} g''(x/(2\sqrt{t})),$$

and hence

$$0 = 2\frac{x}{2\sqrt{t}} g'(x/(2\sqrt{t})) + g''(x/(2\sqrt{t}))$$

so

$$0 = 2pg'(p) + g''(p).$$

(b) Set $h = g'$, then we can solve

$$h'(p) + 2ph(p) = 0$$

by multiplying by the integrating factor e^{p^2}:

$$0 = e^{p^2} h'(p) + 2pe^{p^2} h(p) = \frac{d}{dp}\left(e^{p^2} h(p)\right).$$

Hence $e^{p^2} h(p) = A$ and $h(p) = Ae^{-p^2}$. It follows that

$$g(p) = \int_0^p Ae^{-q^2}\, dq + B$$

and hence

$$u(x,t) = \int_0^{x/(2\sqrt{t})} Ae^{-q^2}\, dq + B$$

The initial condition tells us that

$$\lim_{t\to 0} u(x,t) = \phi(x) = \begin{cases} 1 & \text{if } x \geq 0; \\ 0 & \text{if } x < 0. \end{cases}$$

But

$$\lim_{t\to 0} u(x,t) = \begin{cases} \int_0^{\infty} Ae^{-q^2} dq + B & \text{if } x > 0; \\ \int_0^{-\infty} Ae^{-q^2} dq + B & \text{if } x < 0. \end{cases} = \begin{cases} \frac{A\sqrt{\pi}}{2} + B & \text{if } x > 0; \\ -\frac{A\sqrt{\pi}}{2} + B & \text{if } x < 0. \end{cases}$$

Solving the two equations $(\sqrt{\pi}/2)A + B = 1$ and $-(\sqrt{\pi}/2)A + B = 0$ gives $A = 1/\sqrt{\pi}$ and $B = 1/2$, so

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{x/(2\sqrt{t})} e^{-q^2}\, dq + \frac{1}{2}.$$

(c) Observe that, by the First Fundamental Theorem of Calculus and the chain rule,

$$\partial_x u(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t} = S(x,t).$$

We know that the heat kernel solves the heat equation. Moreover, looking at the graph of $x \mapsto S(x,t)$ for smaller and smaller t we might guess that $x \mapsto S(x,t)$ tends towards a Dirac delta distribution as $t \to 0$. This means that it appears that S solves (7.2) with initial data being the Dirac delta distribution. While our initial data ϕ is not differentiable in the usual sense, we can differentiate it in the sense of distributions and its derivative is the Dirac delta distribution. Thus it makes sense that $\partial_x u(x,t) = S(x,t)$ since they both appear to solve the same initial value problems for the heat equations.