Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions 8

8.1 We recall that in lectures we claimed

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t}\phi(\mathbf{y})d\sigma(\mathbf{y})\right) + \frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t}\psi(\mathbf{y})d\sigma(\mathbf{y}), \tag{6.14}$$

is a solution to (6.10).

(a) We compute that if $\phi(x,y,z) = 0$ and $\psi(x,y,z) = 1$ for all $(x,y,z) \in \mathbf{R}^3$, then according to (6.14)

$$\begin{aligned} u(\mathbf{x},t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} 0\, d\sigma(\mathbf{y})\right) + \frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} 1\, d\sigma(\mathbf{y}) \\ &= 0 + \frac{4\pi t^2}{4\pi t} = t. \end{aligned}$$

(b) Equally, we compute that if $\phi(\mathbf{x}) = 0$ and $\psi(\mathbf{x}) = y$ for all $\mathbf{x} = (x,y,z) \in \mathbf{R}^3$, then according to (6.14)

$$u(\mathbf{x},t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} 0\, d\sigma(\mathbf{y})\right) + \frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} v\, d\sigma(\mathbf{y}) = frac14\pi t\int_{|\mathbf{y}-\mathbf{x}|=t} v\, d\sigma(\mathbf{y})$$

where we write $\mathbf{y} = (u,v,w) \in \mathbf{R}^3$. To simplify the integral above we observe that $\mathbf{y} \mapsto v$ is harmonic, so Theorem 5.6 gives that

$$\frac{1}{4\pi t}\int_{|\mathbf{y}-\mathbf{x}|=t} v\, d\sigma(\mathbf{y}) = t\left(\frac{1}{4\pi t^2}\int_{|\mathbf{y}-\mathbf{x}|=t} v\, d\sigma(\mathbf{y})\right) = ty,$$

so $u(\mathbf{x},t) = ty$ for $\mathbf{x} = (x,y,z) \in \mathbf{R}^3$.

8.2 The formula

$$\begin{aligned} u(x,y,t) = &\frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{a^2+b^2\le t^2}\frac{\phi(a+x,b+y)}{\sqrt{t^2-a^2-b^2}}dadb\right) \\ &+ \frac{1}{2\pi}\int_{a^2+b^2\le t^2}\frac{\psi(a+x,b+y)}{\sqrt{t^2-a^2-b^2}}dadb \end{aligned} \tag{6.16}$$

with $\phi(x,y) = 0$ and $\psi(x,y) = A$ for all $(x,y) \in \mathbf{R}^2$ reads

$$\begin{aligned} u(x,y,t) &= \frac{1}{2\pi}\int_{a^2+b^2\le t^2}\frac{A}{\sqrt{t^2-a^2-b^2}}dadb = \frac{t^2}{2\pi t}\int_{a^2+b^2\le 1}\frac{A}{\sqrt{1-a^2-b^2}}dadb \\ &= \frac{1}{2\pi}\int_0^1\int_0^{2\pi}\frac{A}{\sqrt{1-r^2}}rd\theta dr = t\int_0^1\frac{Ar}{\sqrt{1-r^2}}dr = -At\sqrt{1-r^2}\Big|_0^1 = At \end{aligned}$$

8.3 We apply the change of variables $\mathbf{y} = r\mathbf{z}$ to obtain

$$\overline{u}(\mathbf{x}) = \frac{1}{4\pi r^2}\int_{|\mathbf{y}|=r} u(\mathbf{y})d\sigma(\mathbf{y}) = \frac{1}{4\pi}\int_{|\mathbf{z}|=1} u(r\mathbf{z})d\sigma(\mathbf{z})$$

where $r = |\mathbf{x}|$. Therefore, recalling that the Laplacian in spherical coordinates is

$$\Delta = \Delta_{(r,\theta,\phi)} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2},$$

we see that

$$\Delta\overline{u}(\mathbf{x}) = \frac{1}{4\pi}\int_{|\mathbf{z}|=1}\Delta_{(r,\theta,\phi)}u(r\mathbf{z})d\sigma(\mathbf{z}) = \frac{1}{4\pi}\int_{|\mathbf{z}|=1}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)u(r\mathbf{z})d\sigma(\mathbf{z}). \tag{1}$$

Writing (s,φ,ϑ) as the spherical coordinates of $\mathbf{z}$, we have

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)u(r\mathbf{z})=\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)u(rs,\varphi,\vartheta)=s^2\left(\partial_1^2u(rs,\varphi,\vartheta)+\frac{2}{rs}\partial_1u(rs,\varphi,\vartheta)\right).$$

Since we are integrating on the unit sphere, $s=1$, and so

$$\begin{aligned}&\frac{1}{4\pi}\int_{|\mathbf{z}|=1}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)u(r\mathbf{z})d\sigma(\mathbf{z})\\&=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\partial_1^2u(rs,\varphi,\vartheta)+\frac{2}{rs}\partial_1u(rs,\varphi,\vartheta)\right)\sin\vartheta d\varphi d\vartheta\\&=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial^2u(r,\phi,\theta)}{\partial r^2}+\frac{2}{r}\frac{\partial u(r,\phi,\theta)}{\partial r}\right)\sin\theta d\phi d\theta\\&=\frac{1}{4\pi r^2}\int_{|\mathbf{y}|=r}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)u(\mathbf{y})d\sigma(\mathbf{y}).\end{aligned}\tag{2}$$

Moreover

$$\begin{aligned}&\frac{1}{4\pi r^2}\int_{|\mathbf{y}|=r}\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)u(\mathbf{y})d\sigma(\mathbf{y})\\&=\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}r^2\sin\theta\left(\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)u(r,\phi,\theta)\,d\phi d\theta\\&=\frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}\left(\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{\partial^2}{\partial\phi^2}\right)u(r,\phi,\theta)\,d\phi d\theta=0,\end{aligned}$$

since $u(r,\phi,\theta)$ is 2π-periodic in ϕ and $\sin 0=\sin\pi=0$. Combining this with (1) and (2) we find that $\Delta\overline{u}=\overline{\Delta u}$.

8.4 Since g_{odd} and h_{odd} are odd, we can write (6.8) as

$$\begin{aligned}v(x,t)&=\frac{1}{2}\left(g_{\text{odd}}(x+ct)+g_{\text{odd}}(x-ct)\right)+\frac{1}{2c}\int_{x-ct}^{x+ct}h_{\text{odd}}(y)dy\\&=\frac{1}{2}\left(g_{\text{odd}}(ct+x)-g_{\text{odd}}(ct-x)\right)+\frac{1}{2c}\left(\int_{x-ct}^{ct-x}h_{\text{odd}}(y)dy+\int_{ct-x}^{ct+x}h_{\text{odd}}(y)dy\right)\\&=\frac{\partial}{\partial t}\left(\frac{1}{2c}\int_{ct-x}^{ct+x}g_{\text{odd}}(y)dy\right)+\frac{1}{2c}\int_{ct-x}^{ct+x}h_{\text{odd}}(y)dy\end{aligned}$$

when $0\le x\le ct$.

8.5 (a) We can rewrite the wave equation

$$\partial_t^2v(r,t)-\partial_r^2v(r,t)=0\quad\text{for } r\in\mathbf{R}\text{ and } t>0\tag{3}$$

as the system

$$\begin{cases}\partial_tu(x,t)+\partial_xu(x,t)=0\\\partial_tv(x,t)-\partial_xv(x,t)=u(x,t)\end{cases}$$

Via the method of characteristics, we see that a general solution to the first equation is $u(r,t)=h(r-t)$. We observe that $v(r,t)=g(r-t)$ is a solution to the second equation with $u(r,t)=h(r-t)$ provided $-2g'(r-t)=h(r-t)$. Since h was arbitrary, this says nothing more that g is differentiable. To find a general solution to the second equation we must add an arbitrary solution to the homogeneous equation $\partial_tv(x,t)-\partial_xv(x,t)=0$, which again via the method of characteristics can be seen to be $v(r,t)=f(r-t)$. Thus a general solution to (3) is

$$v(r,t)=f(r-t)+g(r+t)\tag{4}$$

for arbitrary differentiable functions $f\colon\mathbf{R}\to\mathbf{R}$ and $g\colon\mathbf{R}\to\mathbf{R}$.

(b) We can write a radial solution u as $u(\mathbf{x}, t) = u_0(|\mathbf{x}|, t)$ for some $u_0\colon [0, \infty)^2 \to \mathbf{R}$. Furthermore $u_0(r, t) = \overline{u}_\mathbf{0}(r, t)$ for all $r, t > 0$, where $\overline{u}_\mathbf{0}$ is the spherical mean of u about the origin. Therefore $u_0 = \overline{u}_\mathbf{0}$ satisfies (6.12) (with $\mathbf{x} = \mathbf{0}$). By the same argument as in the notes,

$$v(r, t) := r u_0(r, t) \tag{5}$$

satisfies the wave equation (6.13) and $v(0, t) = 0$. Therefore v has the form (4) for some f and g. In order to ensure $v(0, t) = 0$ we choose f and g so that $r \mapsto v(r, t)$ is odd. One way to do this is by choosing f and g so that $f(-x) = -g(x)$ for all $x \in \mathbf{R}$. By (5) we have

$$u(\mathbf{x}, t) = u_0(|\mathbf{x}|, t) = \frac{f(|\mathbf{x}| - t) + g(|\mathbf{x}| + t)}{|\mathbf{x}|}. \tag{$*$}$$

(c) At first sight it appears that $(*)$ may develop a singularity at the origin, as we divide by $|\mathbf{x}|$. However the requirement that $v(0, t) = 0$ above has the potential to mitigates this problem.

It is instructive to test a few sensible examples of f and g to see if it is possible to create a singularity at the origin. Remember we must make sure $r \mapsto f(r - t) + g(r + t)$ is odd! You should find your attempts to create a singularity are always thwarted.

Try, for example, $g(s) = s^m$ for $s \geq 0$ and $m = 2, 3$.

This is a hard problem to answer rigorously, but is a nice exercise to play with to investigate what the truth might be.

Section 2.4.1 of Evans *Partial Differetial Equations* makes concrete statements about the regularity of solutions to the wave equation. See Theorems 1 and 2 there. They state that if the initial data is sufficiently smooth, then the u will be correspondingly smooth. In broad terms there is no difference between one and three dimensions.