Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions to Homework 4

4.1 For two functions $u, v \in C^2(\overline{\Omega})$, Green's first identity (5.8) says

$$\int_{\partial\Omega} v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}) = \int_{\Omega}(\nabla v(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + v(\mathbf{x})\Delta u(\mathbf{x}))d\mathbf{x}.$$

Reversing the roles of u and v, we also have

$$\int_{\partial\Omega} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}) = \int_{\Omega}(\nabla u(\mathbf{x}) \cdot \nabla v(\mathbf{x}) + u(\mathbf{x})\Delta v(\mathbf{x}))d\mathbf{x}.$$

Subtracting the first equality from the second, we obtain

$$\int_{\Omega} u(\mathbf{x})\Delta v(\mathbf{x}) - v(\mathbf{x})\Delta u(\mathbf{x})d\mathbf{x} = \int_{\partial\Omega} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) - v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}).$$

4.2 Let Ω be an open set with C^1 boundary, and let $f\colon \Omega \to \mathbf{R}$ and $g\colon \partial\Omega \to \mathbf{R}$. Suppose we had two solutions $u \in C^2(\Omega)$ to the following boundary value problems.

(a) Suppose we had two solutions $u_1, u_2 \in C^2(\Omega)$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega. \end{cases}$$

Then $v = u_1 - u_2$ solves

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \text{ and} \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, using (5.8),

$$\int_{\Omega} |\nabla v(\mathbf{x})|^2 d\mathbf{x} = \int_{\Omega} \nabla v(\mathbf{x}) \cdot \nabla v(\mathbf{x})d\mathbf{x} = -\int_{\Omega} v(\mathbf{x})\Delta v(\mathbf{x})d\mathbf{x} = 0$$

which implies $\nabla v = 0$, so v must be a constant. However, since v is zero on $\partial\Omega$, it must be that $v = 0$. Therefore $u_1 = u_2$.

(b) Suppose we had two solutions $u_1, u_2 \in C^2(\Omega)$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} + au = h & \text{on } \partial\Omega \end{cases}$$

Then $v = u_1 - u_2$ solves

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial v}{\partial \mathbf{n}} + av = 0 & \text{on } \partial\Omega \end{cases}$$

Therefore, using (5.8),

$$\begin{aligned} 0 \le \int_{\Omega} |\nabla v(\mathbf{x})|^2 d\mathbf{x} = \int_{\Omega} \nabla v(\mathbf{x}) \cdot \nabla v(\mathbf{x})d\mathbf{x} &= -\int_{\Omega} v(\mathbf{x})\Delta v(\mathbf{x})d\mathbf{x} + \int_{\partial\Omega} v(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}) \\ &= \int_{\partial\Omega} v(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x})d\sigma(\mathbf{x}) \\ &= -a\int_{\partial\Omega} |v(\mathbf{x})|^2 d\sigma(\mathbf{x}) \le 0 \end{aligned}$$

which implies $\nabla v = 0$, so v must be a constant. This means $\frac{\partial v}{\partial \mathbf{n}} = 0$ on $\partial\Omega$ and so the boundary condition then tells us that $av = 0$, which implies $v = 0$, since $a > 0$. Alternatively, we also see from the above calculation that

$$\int_{\partial\Omega} |v(\mathbf{x})|^2 d\sigma(\mathbf{x}) = 0,$$

so if v is a constant it must be zero.

In any case, we can thus conclude that $u_1 = u_2$.

4.3 Suppose we had two solutions $u_1, u_2 \in C^2(\Omega)$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega\text{, and} \\ \frac{\partial u}{\partial \mathbf{n}} = h & \text{on } \partial\Omega \end{cases}$$

Then $v = u_1 - u_2$ solves

$$\begin{cases} \Delta v = 0 & \text{in } \Omega\text{, and} \\ \frac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases}$$

Therefore, using (5.8),

$$\begin{aligned} \int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} = \int_\Omega \nabla v(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} &= -\int_\Omega v(\mathbf{x}) \Delta v(\mathbf{x}) d\mathbf{x} + \int_{\partial\Omega} v(\mathbf{x}) \frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) d\sigma(\mathbf{x}) \\ &= \int_{\partial\Omega} v(\mathbf{x}) \frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) d\sigma(\mathbf{x}) = 0 \end{aligned}$$

which implies $\nabla v = 0$, so v must be a constant. This means $u_1 = u_2 + c$ for some constant $c \in \mathbf{R}$, so, no, there cannot be any more $C^2(\overline{\Omega})$ solutions.