Partial Differential Equations (TATA27)
Spring Semester 2019
Solutions to Homework 2

2.1 Using the method of characteristics, we set $s(t) = u(X(t), Y(t))$ where $(X, Y)\colon \mathbf{R} \to \mathbf{R}^2$ are the characteristic curves. If they satisfy the equations

$$\begin{cases} X'(t) = 1 \\ Y'(t) = Y(t) \end{cases}$$

then $s'(t) = u_x(X(t), Y(t)) + Y(t)u_y(X(t), Y(t)) = 0$, so s is a constant function. Solving the ODEs above, we find $X(t) = t + c$ and $Y(t) = Ce^t$ for constants c and C. Thus any solution to the PDE is constant on the lines $y = Ce^x$ and so is of the form

$$u(x, y) = f(ye^{-x}) \tag{1}$$

for an arbitrary function $f\colon [0, \infty) \to \mathbf{R}$. The requirement that $u \in C^1(\mathbf{R}^2_+)$ implies we need f to be continuously differentiable on $(0, \infty)$. If u is to belong to $C(\overline{\mathbf{R}^2_+})$, then in particular

$$u(x, 0) = \lim_{y\to 0} u(x, y).$$

Substituting in the boundary condition $u(x, 0) = \phi(x)$ and (1), we find

$$\phi(x) = u(x, 0) = \lim_{y\to 0} u(x, y) = \lim_{y\to 0} f(ye^{-x}) = f(0).$$

Thus, if the PDE is to have a solution in $C(\overline{\mathbf{R}^2_+}) \cap C^1(\mathbf{R}^2_+)$, ϕ must be a constant function. Consequetly, (a) if $\phi(x) = x$ there are no such solutions, and (b) if $\phi(x) = 1$, then (1) is a solution for any continuous f which is continuously differentiable on $(0, \infty)$ and such that $f(0) = 1$.

2.2 [Olle Abrahamsson] We will show that the equation has more than one solution by solving the equation directly. We multiply the equation by the integrating factor e^x to find

$$f(x)e^x = u''(x)e^x + u'(x)e^x = \frac{d}{dx}\left(u'(x)e^x\right)$$

so

$$u'(x) = e^{-x} \int_0^x f(t)e^t dt + c_0e^{-x}$$

for a constant c_0. Thus

$$\begin{aligned} u(x) &= \int_0^x e^{-s} \int_0^s f(t)e^t dtds - c_0e^{-x} + c_1 \\ &= \int_0^x f(t)(1 - e^{t-x})dt - c_0e^{-x} + c_1, \end{aligned}$$

where c_1 is a constant.

The condition $u'(0) = u(0)$ says that $0+c_0 = 0-c_0+c_1$, so $c_1 = 2c_0$, and $u'(0) = \frac{1}{2}(u'(\ell)+u(\ell))$ says

$$\begin{aligned} c_0 &= \frac{1}{2}\left(e^{-\ell} \int_0^\ell f(t)e^t dt + c_0e^{-\ell} + \int_0^\ell f(t)(1 - e^{t-\ell})dt - c_0e^{-\ell} + 2c_0\right) \\ &= \frac{1}{2}\int_0^\ell f(t)dt + c_0, \end{aligned}$$

so we require $\int_0^\ell f(t)dt = 0$, but no restriction on c_0. Thus we have that

$$u(x) = \int_0^x f(t)(1 - e^{t-x})dt - c_0e^{-x} + 2c_0 \tag{2}$$

is a solution for any $c_0 \in \mathbf{R}$. We can check directly that such a u will be twice continuously differentiable if and only if f is continuous. Therefore we have shown that

(b) For a solution to exist in $C^2([0,\ell])$ we require f to be continuous and $\int_0^\ell f(t)dt = 0$.

(a) If these conditions are satisfied then we have an infinite number of solutions in $C^2([0,\ell])$ given by (2) for an arbitrary $c_0 \in \mathbf{R}$.

2.3 For $\varepsilon > 0$ set $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon|\mathbf{x}|^2$. As the sum of two continuous functions, v is continuous on $\overline{\Omega}$ and so must attain a maximum somewhere in the compact set $\overline{\Omega} = \Omega \cup \partial\Omega$. We will now rule out the possibility that v attains its maximum in Ω. Suppose to the contrary that v attains this maximum $\mathbf{x} \in \Omega$. Then we know $\mathbf{x}$ is a critical point, so $\nabla v(\mathbf{x}) = 0$ and, by the second derivative test, $\Delta v(\mathbf{x}) = \sum_{j=1}^n \partial_j^2 v(\mathbf{x}) \leq 0$. Therefore

$$\Delta v(\mathbf{x}) + \mathbf{x} \cdot \nabla v(\mathbf{x}) = \Delta v(\mathbf{x}) + 0 \leq 0 + 0 = 0.$$

But on the other hand, we can compute

$$\Delta v(\mathbf{x}) + \mathbf{x} \cdot \nabla v(\mathbf{x}) = \Delta u(\mathbf{x}) + \mathbf{x} \cdot \nabla u(\mathbf{x}) + 2\varepsilon|\mathbf{x}|^2 + 2\varepsilon n \geq 2\varepsilon|\mathbf{x}|^2 + 2\varepsilon n > 0,$$

via the differential inequality u satisfies. These two inequalities contradict each other, so v cannot attain its maximum in Ω.

Therefore v must attain its maximum at a point $\mathbf{y} \in \partial\Omega$. Thus, for any $\mathbf{x} \in \overline{\Omega}$,

$$u(\mathbf{x}) \leq v(\mathbf{x}) \leq v(\mathbf{y}) = u(\mathbf{y}) + \varepsilon|\mathbf{y}|^2 \leq u(\mathbf{y}) + \varepsilon C^2 \leq \max_{\partial\Omega} u + \varepsilon C^2,$$

where C is the constant obtained from the fact Ω is bounded. Since the above inequality holds for any $\varepsilon > 0$, we have $u(\mathbf{x}) \leq \max_{\partial\Omega} u$ for any $\mathbf{x} \in \overline{\Omega}$, so

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u$$

Because $\partial\Omega \subseteq \overline{\Omega}$ we have that $\max_{\partial\Omega} u \leq \max_{\overline{\Omega}} u$ and combining these two inequalities we get that $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$ and the maximum of u is attained on $\partial\Omega$.

2.4 (a) We can compute

$$\begin{aligned} \partial_x v &= (\partial_1 u)(x+a, y+b), \\ \partial_y v &= (\partial_2 u)(x+a, y+b) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \partial_x^2 v &= (\partial_1^2 u)(x+a, y+b), \\ \partial_y^2 v &= (\partial_2^2 u)(x+a, y+b). \end{aligned}$$

so

$$\begin{aligned} \Delta v(x,y) &= \partial_x^2 v(x,y) + \partial_y^2 v(x,y) \\ &= (\partial_1^2 u)(x+a, y+b) + (\partial_2^2 u)(x+a, y+b) = \Delta u(x+a, y+b) = 0. \end{aligned}$$

(b) We can compute

$$\begin{aligned} \partial_x w(x,y) &= \cos\alpha(\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad - \sin\alpha(\partial_2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha), \quad \text{and} \\ \partial_x^2 w(x,y) &= \cos^2\alpha(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad - 2\cos\alpha\sin\alpha(\partial_2\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad + \sin^2\alpha(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \end{aligned}$$

And

$$\begin{aligned} \partial_y w(x,y) &= \sin\alpha(\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad + \cos\alpha(\partial_2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha), \quad \text{and} \\ \partial_x^2 w(x,y) &= \sin^2\alpha(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad + 2\cos\alpha\sin\alpha(\partial_2\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \\ &\quad + \cos^2\alpha(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) \end{aligned}$$

so

$$\begin{aligned}\Delta w(x,y) &= \cos^2\alpha(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &\quad - 2\cos\alpha\sin\alpha(\partial_2\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &\quad + \sin^2\alpha(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &\quad + \sin^2\alpha(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &\quad + 2\cos\alpha\sin\alpha(\partial_2\partial_1 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &\quad + \cos^2\alpha(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &= (\cos^2\alpha + \sin^2\alpha)(\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &\quad + (\cos^2\alpha + \sin^2\alpha)(\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &= (\partial_1^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &\quad + (\partial_2^2 u)(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha)\\ &= \Delta w(x\cos\alpha + y\sin\alpha, y\cos\alpha - x\sin\alpha) = 0.\end{aligned}$$

2.5 Fix $N > 0$ and let $B_N = \{\mathbf{x} \in \mathbf{R}^n \mid |\mathbf{x}| < N\}$ be a ball in $\mathbf{R}^n$ centred at the origin. Then

$$\begin{aligned}&\int_{B_N} |u(\mathbf{x},t)|^2 d\mathbf{x} - \int_{B_N} |u(\mathbf{x},t_0)|^2 d\mathbf{x} = \int_{t_0}^t \frac{d}{ds}\left(\int_{B_N} |u(\mathbf{x},s)|^2 d\mathbf{x}\right) ds\\ &= \int_{t_0}^t \int_{B_N} \frac{\partial}{\partial s}\left(|u(\mathbf{x},s)|^2\right) d\mathbf{x}ds = \int_{t_0}^t \int_{B_N} \frac{\partial}{\partial s}\left(u(\mathbf{x},s)\overline{u(\mathbf{x},s)}\right) d\mathbf{x}ds\\ &= \int_{t_0}^t \int_{B_N} \left(\partial_s u(\mathbf{x},s)\overline{u}(\mathbf{x},s) + u(\mathbf{x},s)\partial_s\overline{u}(\mathbf{x},s)\right) d\mathbf{x}ds,\end{aligned} \tag{3}$$

where the second equality (commuting the derivative and integral) is be justified by our assumptions. From the Schrödinger's equations, we have that

$$\frac{\partial u}{\partial s}(\mathbf{x},s) = \frac{i\hbar}{2m}\Delta u(\mathbf{x},s) + \frac{ie^2}{\hbar|\mathbf{x}|}u(\mathbf{x},s)$$

and taking complex conjugates

$$\frac{\partial \overline{u}}{\partial s}(\mathbf{x},s) = -\frac{i\hbar}{2m}\Delta \overline{u}(\mathbf{x},s) - \frac{ie^2}{\hbar|\mathbf{x}|}\overline{u}(\mathbf{x},s).$$

Thus

$$(\partial_s u(\mathbf{x},s))\overline{u}(\mathbf{x},s) + u(\mathbf{x},s)(\partial_s\overline{u}(\mathbf{x},s)) = \frac{i\hbar}{2m}\left(\Delta u(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\Delta\overline{u}(\mathbf{x},s)\right).$$

The divergence theorem tells us

$$\begin{aligned}&\int_{B_N} \left((\partial_s u(\mathbf{x},s))\overline{u}(\mathbf{x},s) + u(\mathbf{x},s)(\partial_s\overline{u}(\mathbf{x},s))\right) d\mathbf{x}\\ &= \int_{B_N} \frac{i\hbar}{2m}\left(\Delta u(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\Delta\overline{u}(\mathbf{x},s)\right) d\mathbf{x}\\ &= -\frac{i\hbar}{2m}\int_{B_N} \left(\nabla u(\mathbf{x},s)\cdot\nabla\overline{u}(\mathbf{x},s) - \nabla u(\mathbf{x},s)\cdot\nabla\overline{u}(\mathbf{x},s)\right) d\mathbf{x}\\ &\quad + \frac{i\hbar}{2m}\int_{\partial B_N} \left(\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\frac{\partial\overline{u}}{\partial\mathbf{n}}(\mathbf{x},s)\right) d\sigma(\mathbf{x})\\ &= \frac{i\hbar}{2m}\int_{\partial B_N} \left(\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\frac{\partial\overline{u}}{\partial\mathbf{n}}(\mathbf{x},s)\right) d\sigma(\mathbf{x}).\end{aligned}$$

Substituting this into (3) and using our assumptions about the decay of u, we find

$$\begin{aligned}&\left|\int_{B_N} |u(\mathbf{x},t)|^2 d\mathbf{x} - \int_{B_N} |u(\mathbf{x},t_0)|^2 d\mathbf{x}\right|\\ &= \left|\frac{i\hbar}{2m}\int_{t_0}^t \int_{\partial B_N} \left(\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x},s)\overline{u}(\mathbf{x},s) - u(\mathbf{x},s)\frac{\partial\overline{u}}{\partial\mathbf{n}}(\mathbf{x},s)\right) d\sigma(\mathbf{x})ds\right|\\ &\leq \frac{C|t-t_0|\hbar}{2m}|\partial B_N|(1+N)^{-2-\varepsilon}\end{aligned}$$

where $|\partial B_N|$ is the area of the set ∂B_N and equals $3\alpha(3)N^2$, where $\alpha(3)$ is the volume of the unit ball in $\mathbf{R}^3$. Thus $|\partial B_N|(1+N)^{-2-\varepsilon} = 3\alpha(3)N^2(1+N)^{-2-\varepsilon} \to 0$ as $N \to \infty$, which proves

$$\int_{\mathbf{R}^3} |u(\mathbf{x}, t)|^2 d\mathbf{x} = \int_{\mathbf{R}^3} |u(\mathbf{x}, t_0)|^2 d\mathbf{x} = 1,$$

as required.