Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 9

Review of previous seminars

In Seminar 12 we sudied Sections 7.1 and 7.2. Questions 9.1 and 9.2 are directly related to these sections. Even question 9.5 is good preparation for the next seminar, although all the remaining questions are most closely connected to Section 7.3.

9.1 Consider the initial boundary value problem

$$\begin{cases} \partial_{tt}v(x,t) - \partial_{xx}v(x,t) = 0 & \text{for } x \in (0,\ell) \text{ and } t \in (0,T], \\ v(x,0) = g(x) \quad \text{and} \quad \partial_t v(x,0) = h(x) & \text{for } x \in [0,\ell], \text{ and} \\ v(0,t) = 0 \quad \text{and} \quad v(\ell,t) = 0 & \text{for } t \in (0,T]. \end{cases} \tag{1}$$

If v solved the heat equation instead of the wave equation, then v would satify a weak maximum principle:

> The maximum value of v over the set $[0,\ell] \times [0,T]$ is attained on the set $D = ([0,\ell] \times \{0\}) \cup (\{0\} \times [0,T)) \cup (\{\ell\} \times [0,T))$.

(a) Draw a picture of the set D.

(b) Find a specific choice of functions g and h, and $T > 0$, together with a solution v to (1), which prove that such a weak maximum principle for the wave equation is false.

9.2 (a) Write down Theorem 7.1 with the words 'maximum value' replaced by 'minimum value'. Now prove this reformulation. You may use Theorem 7.1 in its original form to help you do this.

(b) Prove Theorem 7.3 using only Theorem 7.1 and 9.2(a).

(c) Use Theorem 7.1 and 9.2(a) to prove the following stability result: If u_1 and u_2 both solve (7.1) with the initial conditions ϕ_1 and ϕ_2 and boundary conditions g_1 and g_2, respectively, then

$$\max_{\mathbf{x}\in\overline{\Omega}, t\in[0,T]} |u_2(\mathbf{x},t) - u_1(\mathbf{x},t)| \le \max_{\mathbf{x}\in\overline{\Omega}} |\phi_2(\mathbf{x}) - \phi_1(\mathbf{x})| + \sup_{\mathbf{x}\in\partial\Omega, t\in(0,T]} |g_2(\mathbf{x},t) - g_1(\mathbf{x},t)|.$$

9.3 Recall that the heat kernel $S\colon \mathbf{R} \times (0,\infty) \to \mathbf{R}$ is defined by the formula

$$S(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/4t}.$$

If $I = \int_{-\infty}^{\infty} S(x,t)dx$ we can write

$$I^2 = \left(\int_{-\infty}^{\infty} S(x,t)dx\right)\left(\int_{-\infty}^{\infty} S(y,t)dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(x,t)S(y,t)dxdy$$

Evaluate the repeated integral on the right by changing to polar coordinates. What is the value of $\int_0^\infty S(x,t)dx$ and why?

9.4 (a) Show that the heat equation is a linear equation.

(b) Show that if u is a solution to the heat equation $\partial_t u(x,t) - \partial_{xx}u(x,t) = 0$ then $v(x,t) = u(\sqrt{\alpha}x, \alpha t)$ is also a solution to the heat equation for any fixed $\alpha > 0$.

Group work

9.5 The aim of this question is to solve the initial value problem (7.2) with initial data

$$\phi(x) = \begin{cases} 1 & \text{if } x > 0; \\ \frac{1}{2} & \text{if } x = 0; \\ 0 & \text{if } x < 0. \end{cases}$$

(a) Since the initial data ϕ is invariant under the transformation in (9.4b), we look for a solution which would also be unchanged by this transformation. Namely we look for a solution of the form

$$u(x,t) = g(x/(2\sqrt{t}))$$

for some $g\colon \mathbf{R} \to \mathbf{R}$. (We have inserted a 2 here just to make the following calculation neater[1].) Show that g solves the ordinary differential equation

$$g''(p) + 2pg'(p) = 0.$$

(b) Find the general formula for solutions to the ordinary differential equation above, then use the initial data ϕ to find the particular solution we are looking for.

(c) Observe that $\partial_x u(x,t)$ is equal to $S(x,t)$, the heat kernel. Can you justify this in any way?

[1] If you don't like it, you are welcome to repeat the question with $u(x,t) = g(x/\sqrt{t})$.