Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 6

Review of previous seminar

6.1 Consider the function $\Phi\colon \mathbf{R}^n \setminus \{\mathbf{0}\} \to \mathbf{R}$ defined by

$$\Phi(\mathbf{x}) = \begin{cases} -\frac{1}{2\alpha(2)} \ln|\mathbf{x}| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|\mathbf{x}|^{n-2}} & \text{if } n > 2, \end{cases}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbf{R}^n$. (So, in particular, $\alpha(2) = \pi$ and $\alpha(3) = 4\pi/3$.) The aim of this exercise is to prove some properties of Φ stated in class and fill in the remaining gaps in the proof of Lemma 5.9.

(a) Prove that Φ is harmonic on $\mathbf{R}^n \setminus \{\mathbf{0}\}$.

(b) Consider the domain $B_r(\mathbf{0}) = \{\mathbf{y} \in \mathbf{R}^n \mid |\mathbf{y}| < r\}$. Then the outward unit normal at $\mathbf{x} \in \partial B_r(\mathbf{0})$ is $\mathbf{n}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$. Prove that

$$\frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{x}) = \frac{-1}{n\alpha(n)} \frac{1}{|\mathbf{x}|^{n-1}}$$

for each $n = 1, 2, \ldots$.

6.2 Prove the following lemma, which is a generalisation of Lemma 5.9 that does not assume that u is harmonic.

Lemma. *Let Ω be an open bounded set with C^1 boundary and suppose that $u \in C^2(\overline{\Omega})$ is such that $\Delta u = f$ for some $f \in C(\overline{\Omega})$. Then*

$$u(\mathbf{x}) = \int_{\partial\Omega} \left\{ \Phi(\mathbf{y} - \mathbf{x}) \left(\frac{\partial u}{\partial \mathbf{n}} \right) (\mathbf{y}) - \left(\frac{\partial \Phi}{\partial \mathbf{n}} \right) (\mathbf{y} - \mathbf{x}) u(\mathbf{y}) \right\} d\sigma(\mathbf{y}) - \int_{\Omega} f(\mathbf{y}) \Phi(\mathbf{y} - \mathbf{x}) d\mathbf{y}.$$

for each $\mathbf{x} \in \Omega$.

[Hint: Follow the proof of Lemma 5.9.]

Preparation for the next seminar

In preparation for seminar 7, read through sections 5.4.5 and 5.4.6.

Group work

Try this exercise after seminar 7. Please try to discuss your solution with others taking the course.

6.3 Use the lemma from question 6.2 to prove the following generalisation of Theorem 5.11.

Theorem. *Let $\Omega \subset \mathbf{R}^n$ be an open bounded set with C^2 boundary, and suppose $h \in C^2(\partial\Omega)$ and $f \in C(\overline{\Omega})$. If G is a Green's function for the Laplacian in Ω then the solution of the boundary value problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \text{ and} \\ u = h & \text{on } \partial\Omega, \end{cases} \tag{$*$}$$

is given by

$$u(\mathbf{x}) = -\int_{\partial\Omega} \left(\frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{y}) \right) h(\mathbf{y}) d\sigma(\mathbf{y}) - \int_{\Omega} f(\mathbf{y}) G(\mathbf{x}, \mathbf{y}) d\mathbf{y}.$$

where $(\partial G(\mathbf{x}, \cdot)/\partial \mathbf{n})(\mathbf{y}) := \mathbf{n}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} G(\mathbf{x}, \mathbf{y})$ is the normal derivative of $\mathbf{y} \mapsto G(\mathbf{x}, \mathbf{y})$.

We proved the uniqueness of solutions to $(*)$ in Section 5.2 of our notes, so when we can find a Green's function we have both the existence and uniqueness of solutions to $(*)$.

Extra problem

The following exercise is quite hard, so can be considered a bonus exercise to do it you have some spare time, but it is nevertheless a excellent way to check you have mastered the material we are studying.

6.4 The aim of this question is to prove Theorem 5.12. Let Ω be an open bounded set with C^2 boundary.

(a) In this part of the question we will prove that the Green's function for the Laplacian in Ω is unique. Suppose we have two Green's functions G_1 and G_2 for the Laplacian in Ω.

i. For each fixed $\mathbf{x} \in \Omega$, prove that $\mathbf{y} \mapsto G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y})$ has a continuous extension which belongs to $C^2(\overline{\Omega})$ and is harmonic in Ω.

ii. By considering a boundary value problem that $\mathbf{y} \mapsto G_1(\mathbf{x}, \mathbf{y}) - G_2(\mathbf{x}, \mathbf{y})$ solves, prove that $G_1 = G_2$.

(b) We now wish to prove the Green's function is symmetric, that is $G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \Omega$.

i. Fix $\mathbf{x}, \mathbf{y} \in \Omega$ with $\mathbf{x} \neq \mathbf{y}$ and consider the functions $\mathbf{z} \mapsto u(\mathbf{z}) := G(\mathbf{x}, \mathbf{z})$ and $\mathbf{z} \mapsto v(\mathbf{z}) := G(\mathbf{y}, \mathbf{z})$. Apply Green's second identity (5.10) to u and v in the domain $\Omega_r := \Omega \setminus \overline{(B_r(\mathbf{x}) \cup B_r(\mathbf{y}))}$ for $r > 0$ so small that $\overline{(B_r(\mathbf{x}) \cup B_r(\mathbf{y}))} \subset \Omega$ and $\overline{B_r(\mathbf{x})} \cap \overline{B_r(\mathbf{y})} = \emptyset$ to obtain that

$$\begin{aligned} 0 = &\int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \\ &+ \int_{\partial B_r(\mathbf{y})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}). \end{aligned} \tag{$\dagger$}$$

ii. Using the definition of the Green's function, prove that

$$\int_{\partial B_r(\mathbf{x})} (G(\mathbf{x}, \mathbf{z}) - \Phi(\mathbf{z} - \mathbf{x})) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \to 0$$

and

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{y}, \mathbf{z}) \left(\frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z} - \mathbf{x}) \right) d\sigma(\mathbf{z}) \to 0$$

as $r \to 0$.

iii. Using the same ideas as in the proof of Lemma 5.9 prove that

$$\int_{\partial B_r(\mathbf{x})} \Phi(\mathbf{z} - \mathbf{x}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) = 0$$

and

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{y}, \mathbf{z}) \frac{\partial \Phi}{\partial \mathbf{n}}(\mathbf{z} - \mathbf{x}) d\sigma(\mathbf{z}) = -G(\mathbf{y}, \mathbf{x}).$$

iv. Combine the results above to show that

$$\int_{\partial B_r(\mathbf{x})} G(\mathbf{x}, \mathbf{z}) \frac{\partial G(\mathbf{y}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) - G(\mathbf{y}, \mathbf{z}) \frac{\partial G(\mathbf{x}, \cdot)}{\partial \mathbf{n}}(\mathbf{z}) d\sigma(\mathbf{z}) \to G(\mathbf{y}, \mathbf{x}). \tag{$\ddagger$}$$

as $r \to 0$. (Observe the left-hand side of (‡) is the first term on the right-hand side of (†).)

v. Swap the roles of $\mathbf{x}$ and $\mathbf{y}$ in (‡) to conclude a similar statement for the second term on the right-hand side of (†). Combine your answer with (†) and (‡) to prove $G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x})$.