Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 5

Preparation for the next seminar

I didn't go over all the material I planned to in the last seminar, so this weeks homework is a bit shorter than normal. First take a quick second look at Sections 5.4.2 and 5.4.4 from last time. Then read carefully through Section 5.4.3 (which was not part of last weeks homework), and attempt the following problem.

5.1 Let Ω be an open set with C^1 boundary. For $\lambda \geq 0$, define the energy of each continuously differentiable $v\colon \overline{\Omega} \to \mathbf{R}$ to be

$$E_\lambda[v] = \frac{1}{2}\int_\Omega (|\nabla v(\mathbf{x})|^2 + \lambda |v(\mathbf{x})|^2) d\mathbf{x}.$$

Show that a function $u \in C^2(\overline{\Omega})$ which satisfies $\Delta u - \lambda u = 0$ in Ω is such that

$$E_\lambda[u] \leq E_\lambda[v]$$

for all $v \in C^1(\overline{\Omega})$ such that $v(\mathbf{x}) = u(\mathbf{x})$ for all $\mathbf{x} \in \partial\Omega$.

Observe that the energy $E_\lambda[v]$ makes sense for functions in $C^1(\overline{\Omega})$, but (assuming a solution to the corresponding boundary value problem exists) a minimiser can sometimes be found in a better class. For example, if $\lambda = 0$, Lemma 5.5 tells us any solution u is smooth in Ω.

Group work

You should work on the following problem after Seminar 6, and then we will discuss possible solutions together in Seminar 7.

5.2 Let Ω be an open set with C^1 boundary and $h\colon \partial\Omega \to \mathbf{R}$ a C^1 function. Define the energy of each continuously differentiable $v\colon \Omega \to \mathbf{R}$ to be

$$E_h[v] = \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x}) v(\mathbf{x}) d\sigma(\mathbf{x}).$$

Show that a function $u \in C^2(\overline{\Omega})$ which satisfies the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} := \mathbf{n}\cdot\nabla u = h & \text{on } \partial\Omega \end{cases}$$

is such that

$$E_h[u] \leq E_h[v]$$

for all $v \in C^1(\overline{\Omega})$. Here $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

Here, in contrast to question 5.1, the boundary condition $\partial u/\partial\mathbf{n} = h$ is incorporated into the energy and we see that a solution u is a minimum of E_h over all $v \in C^1(\overline{\Omega})$ regardless of how v behaves at the boundary.