Partial Differential Equations (TATA27)
Spring Semester 2019
Homework 4

Preparation for the next seminar

In preparation for Seminar 5 read through Sections 5.4.1, 5.4.2 and 5.4.4, and attempt the following problem.

4.1 Green's second identity says that for two functions $u, v \in C^2(\overline{\Omega})$

$$\int_{\Omega} u(\mathbf{x})\Delta v(\mathbf{x}) - v(\mathbf{x})\Delta u(\mathbf{x}) d\mathbf{x} = \int_{\partial\Omega} u(\mathbf{x})\frac{\partial v}{\partial \mathbf{n}}(\mathbf{x}) - v(\mathbf{x})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) d\sigma(\mathbf{x}). \tag{5.10}$$

Use Green's first identity (5.8) to prove (5.10).

Group work

You should work on the following problem after Seminar 5, and then we will discuss possible solutions together in Seminar 6.

4.2 Let Ω be an open set with C^1 boundary, and let $f\colon \Omega \to \mathbf{R}$ and $h\colon \partial\Omega \to \mathbf{R}$. Use Green's first identity (5.8) to prove the uniqueness of solutions $u \in C^2(\overline{\Omega})$ to the following boundary value problems.

(a)

$$\begin{cases} \Delta u = f & \text{in } \Omega \text{, and} \\ u = h & \text{on } \partial\Omega \end{cases}$$

(b)

$$\begin{cases} \Delta u = f & \text{in } \Omega \text{, and} \\ \frac{\partial u}{\partial \mathbf{n}} + au = h & \text{on } \partial\Omega \end{cases}$$

where $\partial u/\partial \mathbf{n} := \mathbf{n} \cdot \nabla u$, $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ and $a > 0$ is a constant.

Review exercises

Heres an additional exercise for you to try.

4.3 Consider a solution $u \in C^2(\overline{\Omega})$ to the boundary value problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \text{, and} \\ \frac{\partial u}{\partial \mathbf{n}} = h & \text{on } \partial\Omega \end{cases}$$

Observe that for any $c \in \mathbf{R}$ $u + c$ is also a solution. Could there be any more $C^2(\overline{\Omega})$ solutions?