

### Partial Differential Equations

**Comment:** All definitions and theorems have been given in lectures. The remaining content is similar to given homework questions or is work done in lectures.

1. Use the method of characteristics to find a smooth function  $u: \mathbf{R}^2 \rightarrow \mathbf{R}$  which solves the equation

$$xu_x(x, t) + u_t(x, t) + 2u(x, t) = 0 \quad \text{for all } (x, t) \in \mathbf{R}^2$$

and satisfies the condition  $u(x, 0) = \ln(1 + x^2)$  for all  $x \in \mathbf{R}$ . [16 marks]

**Solution:**

We search for appropriate curves  $(X, T)$  such that the solution on the curves  $s \mapsto z(s) := u(X(s), T(s))$  behaves nicely. We have

$$z'(s) = \frac{d}{ds}u(X(s), T(s)) = X'(s)\partial_1 u(X(s), T(s)) + T'(s)\partial_2 u(X(s), T(s)),$$

so it seems reasonable to set  $X'(s) = X(s)$  and  $T'(s) = 1$ . Thus  $X(s) = c_X e^s$  and  $T(s) = s + c_T$  for constants  $c_X, c_T \in \mathbf{R}$ . We can then rewrite the PDE as

$$z'(s) + 2z(s) = X(s)\partial_1 u(X(s), T(s)) + \partial_2 u(X(s), T(s)) + 2u(X(s), T(s)) = 0.$$

This is an ODE with general solution  $z(s) = Ae^{-2s}$  for arbitrary  $A \in \mathbf{R}$ .

Now fix  $(x, t)$ . If we choose  $c_X = x$  and  $c_T = t$ , then  $X(s) = xe^s$  and  $T(s) = s + t$ , and when  $s = 0$  the characteristic curve passes through  $(X(0), T(0)) = (x, t)$  and when  $s = -t$  the curve passes through  $(X(-t), T(-t)) = (xe^{-t}, 0)$ . When  $s = -t$  we can use the initial condition to find the value of  $z$ :

$$z(-t) = u(X(-t), T(-t)) = u(xe^{-t}, 0) = \ln(1 + (xe^{-t})^2).$$

But on the other hand, using the form of the general solution to the characteristic ODE,  $z(-t) = Ae^{2t}$ , so  $A = \ln(1 + (xe^{-t})^2)e^{-2t}$ . Equally, for  $s = 0$ ,

$$u(x, t) = z(0) = Ae^{-2 \times 0} = A = e^{-2t} \ln(1 + (xe^{-t})^2),$$

which gives us an expression for the solution  $u$ .

2. Let  $\Omega \subset \mathbf{R}^n$  be a bounded open set and  $\mathbf{b} \in \mathbf{R}^n$  be a vector which satisfies  $\mathbf{b} \cdot \mathbf{x} + n > 0$  for all  $\mathbf{x} \in \Omega$ .

(a) Prove that continuous functions  $u: \overline{\Omega} \rightarrow \mathbf{R}$  which solve

$$\Delta u(\mathbf{x}) + \mathbf{b} \cdot \nabla u(\mathbf{x}) = 0$$

for  $\mathbf{x} \in \Omega$  satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

[Hint: The function  $\mathbf{x} \mapsto \varepsilon|\mathbf{x}|^2$  for  $\varepsilon > 0$  may be useful.] [10 marks]

- (b) Suppose a continuous function  $g: \partial\Omega \rightarrow \mathbf{R}$  is given. Prove that there cannot exist more than one continuous function  $u: \overline{\Omega} \rightarrow \mathbf{R}$  which solves the boundary value problem

$$\begin{cases} \Delta u + \mathbf{b} \cdot \nabla u = 0 & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega. \end{cases}$$

[6 marks]

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**Solution:**

For  $\varepsilon > 0$  set  $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon|\mathbf{x}|^2$ . As the sum of two continuous functions,  $v$  is continuous on  $\overline{\Omega}$  and so must attain a maximum somewhere in the compact set  $\overline{\Omega} = \Omega \cup \partial\Omega$ . We will now rule out the possibility that  $v$  attains its maximum in  $\Omega$ . Suppose to the contrary that  $v$  attains this maximum  $\mathbf{x} \in \Omega$ . Then we know  $\mathbf{x}$  is a critical point, so  $\nabla v(\mathbf{x}) = 0$  and, by the second derivative test,  $\Delta v(\mathbf{x}) = \sum_{j=1}^n \partial_j^2 v(\mathbf{x}) \leq 0$ . Therefore

$$\Delta v(\mathbf{x}) + \mathbf{b} \cdot \nabla v(\mathbf{x}) = \Delta v(\mathbf{x}) + 0 \leq 0 + 0 = 0.$$

But on the other hand, we can compute

$$\Delta v(\mathbf{x}) + \mathbf{b} \cdot \nabla v(\mathbf{x}) = \Delta u(\mathbf{x}) + \mathbf{b} \cdot \nabla u(\mathbf{x}) + 2\varepsilon n + 2\varepsilon \mathbf{b} \cdot \mathbf{x} \geq 2\varepsilon(\mathbf{b} \cdot \mathbf{x} + n) > 0,$$

via the differential equality  $u$  satisfies and the condition on  $\mathbf{b}$ . These two inequalities contradict each other, so  $v$  cannot attain its maximum in  $\Omega$ .

Therefore  $v$  must attain its maximum at a point  $\mathbf{y} \in \partial\Omega$ . Thus, for any  $\mathbf{x} \in \overline{\Omega}$ ,

$$u(\mathbf{x}) \leq v(\mathbf{x}) \leq v(\mathbf{y}) = u(\mathbf{y}) + \varepsilon|\mathbf{y}|^2 \leq u(\mathbf{y}) + \varepsilon C^2 \leq \max_{\partial\Omega} u + \varepsilon C^2,$$

where  $C$  is the constant obtained from the fact  $\Omega$  is bounded. Since the above inequality holds for any  $\varepsilon > 0$ , we have  $u(\mathbf{x}) \leq \max_{\partial\Omega} u$  for any  $\mathbf{x} \in \overline{\Omega}$ , so

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u$$

Because  $\partial\Omega \subseteq \overline{\Omega}$  we have that  $\max_{\partial\Omega} u \leq \max_{\overline{\Omega}} u$  and combining these two inequalities we get that  $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$  and the maximum of  $u$  is attained on  $\partial\Omega$ .

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**3.** Let  $\Omega$  be an open set and  $\phi: \Omega \rightarrow \mathbf{R}$ . Consider the initial boundary value problem

$$\begin{cases} \partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = 0 & \text{for } \mathbf{x} \in \Omega \text{ and } t \in (0, T]; \\ u(\mathbf{x}, 0) = \phi(x) & \text{for } \mathbf{x} \in \Omega; \text{ and} \\ u(\mathbf{y}, t) = 0 & \text{for } \mathbf{y} \in \partial\Omega \text{ and } t \in (0, T]. \end{cases} \quad (\spadesuit)$$

(a) Show that

$$\int_{\Omega} |u(\mathbf{x}, t)|^2 d\mathbf{x}$$

is a decreasing function of  $t$  for each  $u \in C^2(\bar{\Omega} \times [0, T])$  which solves  $(\spadesuit)$ .  
[8 marks]

(b) Use (a) to prove there cannot exist more than one function  $u \in C^2(\bar{\Omega} \times [0, T])$  which solves  $(\spadesuit)$ .  
[8 marks]

**Solution:**

(a) Using  $(\spadesuit)$  and Green's first identity

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} |u(\mathbf{x}, t)|^2 d\mathbf{x} &= \int_{\Omega} u(\mathbf{x}, t) \partial_t u(\mathbf{x}, t) d\mathbf{x} = \int_{\Omega} u(\mathbf{x}, t) \Delta u(\mathbf{x}, t) d\mathbf{x} \\ &= - \int_{\Omega} \nabla u(\mathbf{x}, t) \nabla u(\mathbf{x}, t) d\mathbf{x} = - \int_{\Omega} |\nabla u(\mathbf{x}, t)|^2 d\mathbf{x} \leq 0, \end{aligned}$$

so  $t \mapsto \int_{\Omega} |u(\mathbf{x}, t)|^2 d\mathbf{x}$  is decreasing.

(b) In particular (a) says

$$0 \leq \int_{\mathbf{R}^n} |u(\mathbf{x}, t)|^2 d\mathbf{x} \leq \int_{\mathbf{R}^n} |\phi(\mathbf{x})|^2 d\mathbf{x}$$

for all  $t \in [0, T]$ . This means that if we had two solutions  $u_1$  and  $u_2$  of  $(\spadesuit)$ , then their difference  $w = u_2 - u_1$  would satisfy  $(\spadesuit)$  with  $\phi = 0$ , so

$$0 \leq \int_{\mathbf{R}^n} |u_2(\mathbf{x}, t) - u_1(\mathbf{x}, t)|^2 d\mathbf{x} = \int_{\mathbf{R}^n} |w(\mathbf{x}, t)|^2 d\mathbf{x} \leq 0.$$

Whence  $u_1 = u_2$ .

**4.** For a smooth solution  $u$  of the wave equation  $\partial_{tt}u(x, t) - \partial_{xx}u(x, t) = 0$ , the energy density is defined to be

$$e(x, t) = \frac{1}{2}((\partial_t u(x, t))^2 + (\partial_x u(x, t))^2)$$

and the momentum density

$$p(x, t) = \partial_t u(x, t) \partial_x u(x, t).$$

(a) Show that  $\partial e / \partial t = \partial p / \partial x$  and  $\partial p / \partial t = \partial e / \partial x$ .  
[8 marks]

(b) Show that  $e$  and  $p$  also satisfy the wave equation.  
[8 marks]

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**Solution:**

(a) We compute

$$\begin{aligned}\frac{\partial e}{\partial t}(x, t) &= \partial_t u(x, t) \partial_{tt} u(x, t) + \partial_x u(x, t) \partial_{xt} u(x, t) \\ \frac{\partial e}{\partial x}(x, t) &= \partial_t u(x, t) \partial_{tx} u(x, t) + \partial_x u(x, t) \partial_{xx} u(x, t) \\ \frac{\partial p}{\partial t}(x, t) &= \partial_{tt} u(x, t) \partial_x u(x, t) + \partial_t u(x, t) \partial_{xt} u(x, t) \\ \frac{\partial p}{\partial x}(x, t) &= \partial_{tx} u(x, t) \partial_x u(x, t) + \partial_t u(x, t) \partial_{xx} u(x, t)\end{aligned}$$

So clearly if  $\partial_{tt} u - \partial_{xx} u = 0$  then  $\partial e / \partial t = \partial p / \partial x$  and  $\partial p / \partial t = \partial e / \partial x$ .

(b) Furthermore, using the above results, if  $\partial_{tt} u - \partial_{xx} u = 0$  we have

$$\begin{aligned}\frac{\partial^2 e}{\partial t^2}(x, t) &= \frac{\partial^2 p}{\partial t \partial x}(x, t) = \frac{\partial^2 p}{\partial x \partial t}(x, t) = \frac{\partial^2 e}{\partial x^2}(x, t) \quad \text{and} \\ \frac{\partial^2 p}{\partial t^2}(x, t) &= \frac{\partial^2 e}{\partial t \partial x}(x, t) = \frac{\partial^2 e}{\partial x \partial t}(x, t) = \frac{\partial^2 p}{\partial x^2}(x, t).\end{aligned}$$

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5. Suppose that a solution  $u$  to the Schrödinger equation

$$-i\partial_t u(x, t) = \partial_{xx} u(x, t) - x^2 u(x, t)$$

is of the form  $u(x, t) = T(t)v(x)$ .

(a) Show that  $v$  satisfies the equation

$$v''(x) + (\lambda - x^2)v(x) = 0, \quad (\heartsuit)$$

for some constant  $\lambda$ .

[6 marks]

(b) We saw in lectures that, by performing the substitution  $v(x) = w(x)e^{x^2/2}$ , it is possible to show  $(\heartsuit)$  is equivalent to

$$w''(x) - 2xw'(x) + (\lambda - 1)w(x) = 0. \quad (\diamond)$$

Show that if  $w$  is a power series, that is  $w(x) = \sum_{k=0}^{\infty} a_k x^k$ , then we must have

$$(k+2)(k+1)a_{k+2} = (2k+1-\lambda)a_k \quad \text{for each } k.$$

[6 marks]

(c) Find a polynomial solution  $w$  to  $(\diamond)$  when  $\lambda = 9$ .

[4 marks]

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**Solution:**

- (a) Substituting  $u(x, t) = T(t)v(x)$  in Schrödinger's equation and dividing by  $T(t)v(x)$  gives

$$-i \frac{T'(t)}{T(t)} = \frac{v''(x) - x^2}{v(x)}$$

Since the left-hand side only depends on  $t$  and the right-hand side only depends on  $x$ , both must be equal to  $-\lambda$ , say, for some constant  $\lambda$ . Therefore

$$v''(x) + (\lambda - x^2)v(x) = 0,$$

for some constant  $\lambda$ .

- (b) Substituting this power series in  $(\diamond)$  we get

$$\sum_{k=0}^{\infty} k(k-1)a_k x^{k-2} - \sum_{k=0}^{\infty} 2ka_k x^k + (\lambda - 1) \sum_{k=0}^{\infty} a_k x^k = 0.$$

Equating powers of  $x$  we see a power series solution must satisfy

$$(k+2)(k+1)a_{k+2} = (2k+1-\lambda)a_k \quad \text{for each } k$$

and  $a_0$  and  $a_1$  can be chosen arbitrarily.

- (c) A calculation using the recursion relation from (b) shows that up to a multiplicative constant  $w(x) = 16x^4 - 48x^2 + 12$ .
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