Instructions: Please attempt all questions. You may answer either in English or Swedish. There are five questions, each worth 16 points. To obtain a grade 3, 4 or 5, you must obtain at least 40, 48 or 56 points (50%, 60% or 70%) respectively. You may not use any notes, textbooks or electronic devices. Good luck!

Svara på alla uppgifter. Du får svara antingen på engelska eller svenska. Det finns fem uppgifter och varje uppgift kan ge maximalt 16 poäng. För att få betyg 3, 4 eller 5 krävs minst 40, 48 respektive 56 poäng (50%, 60% respektive 70%). Inga hälpmedel tillåtna. Lycka till!

(1) Use the method of characteristics to find a smooth function $u\colon \mathbf{R}^2 \to \mathbf{R}$ which solves the equation

$$xu_x(x,t) + u_t(x,t) + 8u(x,t) = 0 \quad \text{for all } (x,t) \in \mathbf{R}^2$$

and satisfies the condition $u(x,0) = (x+2)/(1+x^2)$ for all $x \in \mathbf{R}$. **[16 marks]**

(2) Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and $\mathbf{b} \in \mathbf{R}^n$ be a vector which satisfies $\mathbf{b}\cdot\mathbf{x}+n > 0$ for all $\mathbf{x} \in \Omega$.

(a) Prove that continuous functions $u\colon \overline{\Omega} \to \mathbf{R}$ which solve

$$\Delta u(\mathbf{x}) + \mathbf{b} \cdot \nabla u(\mathbf{x}) = 0$$

for $\mathbf{x} \in \Omega$ satisfy the weak maximum principle:

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

[Hint: The function $\mathbf{x} \mapsto \varepsilon|\mathbf{x}|^2$ for $\varepsilon > 0$ may be useful.] **[10 marks]**

(b) Suppose a continuous function $g\colon \partial\Omega \to \mathbf{R}$ is given. Prove that there cannot exist more than one continuous function $u\colon \overline{\Omega} \to \mathbf{R}$ which solves the boundary value problem

$$\begin{cases} \Delta u + \mathbf{b} \cdot \nabla u = 0 & \text{in } \Omega; \\ u = g & \text{on } \partial\Omega. \end{cases}$$

[6 marks]

(3) Consider a solution u to the *damped string equation*

$$\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) + r\partial_t u(x,t) = 0 \quad (x \in \mathbf{R},\ t > 0)$$

for $c^2 = T/\rho$ and given constants $T, \rho, r > 0$. Define the energy of a solution u at time t by the formula

$$E[u](t) = \frac{1}{2}\int_{-\infty}^{\infty} \rho(\partial_t u(x,t))^2 + T(\partial_x u(x,t))^2 dx.$$

(a) Assuming u and its derivatives are sufficiently smooth and tend to zero as $x \to \pm\infty$, show that the energy $E[u]$ is a non-increasing function. **[8 marks]**

(b) Prove that there cannot exist more than one solution u to the damped string equation which satisfies the same assumptions you made in (a) together with the initial conditions $u(x, 0) = f(x)$ and $\partial_t u(x, 0) = g(x)$, for given smooth functions f and g. **[8 marks]**

(4) Let Ω be an open set with C^1 boundary and $h\colon \partial\Omega \to \mathbf{R}$ a C^1 function. Define the energy of each continuously differentiable $v\colon \Omega \to \mathbf{R}$ to be

$$E_h[v] = \frac{1}{2}\int_\Omega |\nabla v(\mathbf{x})|^2 d\mathbf{x} - \int_{\partial\Omega} h(\mathbf{x})v(\mathbf{x})d\sigma(\mathbf{x}).$$

Show that a function $u \in C^2(\overline{\Omega})$ which satisfies the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \text{ and} \\ \frac{\partial u}{\partial \mathbf{n}} := \mathbf{n} \cdot \nabla u = h & \text{on } \partial\Omega \end{cases}$$

is such that

$$E_h[u] \leq E_h[v]$$

for all $v \in C^1(\overline{\Omega})$. Here $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. **[16 marks]**

(5) Suppose that a solution u to the Schrödinger equation

$$-i\partial_t u(x, t) = \partial_{xx} u(x, t) - x^2 u(x, t)$$

is of the form $u(x, t) = T(t)v(x)$.

(a) Show that v satisfies the equation

$$v''(x) + (\lambda - x^2)v(x) = 0, \qquad (\heartsuit)$$

for some constant λ. **[6 marks]**

(b) We saw in lectures that, by performing the substitution $v(x) = w(x)e^{x^2/2}$, it is possible to show ($\heartsuit$) is equivalent to

$$w''(x) - 2xw'(x) + (\lambda - 1)w(x) = 0. \qquad (\diamondsuit)$$

Show that if w is a power series, that is $w(x) = \sum_{k=0}^{\infty} a_k x^k$, then we must have

$$(k + 2)(k + 1)a_{k+2} = (2k + 1 - \lambda)a_k \quad \text{for each } k.$$

[6 marks]

(c) Find a polynomial solution w to ($\diamondsuit$) when $\lambda = 9$. **[4 marks]**