Instructions: Please attempt all questions. You may answer either in English or Swedish. There are five questions, each worth 16 points. To obtain a grade 3, 4 or 5, you must obtain at least 40, 48 or 56 points (50%, 60% or 70%) respectively. You may not use any notes, textbooks or electronic devices. Good luck!

Svara på alla uppgifter. Du får svara antingen på engelska eller svenska. Det finns fem uppgifter och varje uppgift kan ge maximalt 16 poäng. För att få betyg 3, 4 eller 5 krävs minst 40, 48 respektive 56 poäng (50%, 60% respektive 70%). Inga hälpmedel tillåtna. Lycka till!

(1) One day Baby Elwin and his brother Arthur go on a trip to the science museum *Fenomenmagasinet* with their father. They play with a model the Göta Canal and Arthur suggests the following first order PDE as a model for shallow water waves along a thin canal:

$$u_t(x,t) + u(x,t)u_x(x,t) = 0 \quad \text{for all } x \in \mathbf{R} \text{ and } t > 0.$$

Here $u(x,t)$ is the height of the water at time t and a distance x along the canal.

Although the equation is nonlinear the method of characteristics can still be used to construct solutions.

(a) Show that this method leads to the system of ODEs

$$\begin{cases} X'(s) = z(s) \\ T'(s) = 1 \\ z'(s) = 0 \end{cases}$$

where $s \mapsto (X(s), T(s))$ is a characteristic curve in the (x,t)-plane and $z(s) = u(X(s), T(s))$. **[4 marks]**

(b) Motivate why the characteristic curves must be straight lines. **[6 marks]**

(c) Sketch some typical characteristic curves $s \mapsto (X(s), T(s))$ when the solution u satisfies the initial condition

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0; \\ x & \text{if } x \in [0,1]; \\ 1 & \text{if } x > 1. \end{cases}$$

[6 marks]

(2) It's breakfast time for Baby Elwin and his father is warming up his porridge in a saucepan on the hob. Denote the interior of the volume occupied by the porridge by Ω and its boundary by $\partial\Omega$. Neglecting the effects of convection and stirring, the temperature of the porridge u is modelled by the heat equation

$$\partial_t u(\mathbf{x},t) - \Delta u(\mathbf{x},t) = 0$$

for $\mathbf{x} \in \Omega$ and $0 < t \leq T$. The initial temperature of the porridge $u(\mathbf{x},0)$ ($\mathbf{x} \in \Omega$) is known and Elwin's dad can control the boundary values $u(\mathbf{x},t)$ for $\mathbf{x} \in \partial\Omega$ and $t > 0$.

(a) Assuming that Ω is a bounded domain and the temperature u is a continuous function in $\overline{\Omega} \times [0, T]$, prove that the highest temperature of the porridge occurs at a point $(\mathbf{x}, t) \in \overline{\Omega} \times [0, T]$ for which either $t = 0$ or $\mathbf{x} \in \partial\Omega$. (That is, prove the weak maximum principle for the heat equation.) **[8 marks]**

If Baby Elwin's dad instead uses the microwave to heat up the porridge the model is modified to the inhomogeneous heat equation

$$\partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) = F(\mathbf{x}, t) \quad (\mathbf{x} \in \Omega \text{ and } 0 < t \leq T)$$

where $F(\mathbf{x}, t)$ is controlled by Elwin's dad, but the boundary values of u are now fixed at room temperature.

(b) Do solutions u to the inhomogeneous equation satisfy the same maximum principle from (a) for all possible choices of F? Motivate your answer. **[5 marks]**

(c) On the basis of the phenomena discussed above, which method of heating porridge would you recommend for fathers looking after their children. Briefly motivate your answer. **[3 marks]**

(3) Baby Elwin has a toy guitar which both he and his brother Arthur enjoy playing with. One day, while their father is looking after them, Arthur suggests the displacement of the string at a point x along the string's length ℓ at time t is modelled by a function $u(x, t)$ which satisfies the *damped string equation*

$$\partial_{tt} u(x, t) - c^2 \partial_{xx} u(x, t) + r \partial_t u(x, t) = 0 \quad (x \in (0, \ell),\ t > 0)$$

for given constants $c, r > 0$ and the boundary conditions

$$u(0, t) = u(\ell, t) = 0$$

for all $t > 0$.

(a) Show that if we look for a solution of the form $u(x, t) = X(x)T(t)$ (that is, we separate variables) the functions X and T must satisfy

$$c^2 X''(x) + \lambda X(x) = 0$$

and

$$T''(t) + rT(t) + \lambda T(t) = 0$$

respectively, for some constant λ. **[4 marks]**

(b) Show that the boundary conditions imply λ must be taken to be positive if we wish to find a non-zero solution via this method. Calculate the precise values λ can take on. **[4 marks]**

(c) Show that the function T obtained via this method decays exponentially in t provided the damping coefficient r satisfies $r^2/4 > \lambda$. **[4 marks]**

(d) Without doing any further precise calculations, describe how the solution will decay if $r^2/4 < \lambda$. **[4 marks]**

(4) Baby Elwin and his brother, Arthur, are playing on an old trampoline. The frame is circular in shape, but, due to the unevenness of the ground beneath it, it has bent so that its height varies around the edge of the trampoline. Arthur produces a simple model for the stationary taut trampoline before they climb on it: The height of the trampoline above a point $\mathbf{x}$ on the ground is given by the solution $u(\mathbf{x})$ of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } D_a \text{, and} \\ u = h & \text{on } \partial D_a, \end{cases}$$

where $D_a = \{\mathbf{x} \in \mathbf{R}^2 \mid |\mathbf{x}| < a\}$ is the disc of radius a above which the trampoline lies.

Arthur even manages to derive the Poisson formula for the solution u:

$$u(\mathbf{x}) = \frac{(a^2 - |\mathbf{x}|^2)}{2\pi a} \int_{|\mathbf{y}|=a} \frac{h(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} d\sigma(\mathbf{y}).$$

(a) For a harmonic function $u\colon \Omega \to \mathbf{R}$, where $\Omega \subset \mathbf{R}^2$ is an open set, state the mean value property u satisfies. **[4 marks]**

(b) Use the Poisson formula above to prove the mean value property you stated in part (a). **[6 marks]**

(c) Suppose $u\colon \overline{D_a} \to \mathbf{R}$ is harmonic in D_a and such that $u(a, \theta) = \sin(4\theta) + 2$ on ∂D_a (written in polar coordinates). Without calculating the solution explicitly, compute the maximum value of u in $\overline{B_a}$ and the value of u at the origin. **[6 marks]**

(5) Baby Elwin has trouble understanding derivatives, so prefers to use finite differences instead, saying that he heard somewhere they are more or less the same. The aim of this question is to convince his brother, Arthur, that, in some sense, Baby Elwin is justified in conflating the two notions.

Fix $\delta x > 0$. For a function $u\colon \mathbf{R} \to \mathbf{R}$ which is three times continuously differentiable, define $x_j = j\delta x$ and set $u_j = u(x_j)$.

(a) Define the forward, backward and centred difference of u at x_j. **[6 marks]**

(b) Use Taylor's theorem to estimate the error between the differences you defined above and the first derivative $u'(x_j)$. State clearly any assumptions you make on the function u. **[10 marks]**