Graduate Course in Partial Differential Equations (MAI0133)
Spring Semester 2017
Homework 3

3.1 Assume U is an open connected set with smooth boundary. Furthermore suppose $a^{ij}, b^i, c \in L^\infty(U)$ for all $i,j = 1,\ldots,n$ and $(a^{ij})_{ij}$ satisfies the ellipticity condition $\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$ for some $\theta > 0$ and all $\xi = (\xi_1,\ldots,\xi_n)$ and x in U. Let

$$Lu = -\sum_{i,j=1}^n (a^{ij}u_{x_i})_{x_j} + \sum_{i=1}^n b^i u_{x_i} + cu.$$

Prove that there exists a constant C depending only on θ, $\|b^i\|_{L^\infty}$ and $\|c\|_{L^\infty}$ such that a weak solution $u \in H_0^1(U)$ to $Lu = f$ satisfies

$$\|Du\|_{L^2(U)} \leq C\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right).$$

3.2 Assume U is an open connected set. Suppose $v, w \in L^2(U)$ and are supported in a set V which is compactly contained in U. Prove that

$$\int_U v(x)D_k^{-h}w(x)dx = -\int_U D_k^h v(x)w(x)dx$$

and

$$D_k^h(vw) = v^h D_k^h w + wD_k^h v,$$

for $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U)$ where $v^h(x) = v(x + he_k)$.

Please feel free to look through Chapter 5 and 6 in Evans for supporting theory and further reading. I am happy to also discuss your thoughts/problems related to the theory or any other questions you decide to think about related to the course.