

Geometry
Second Semester 2010/11
 Homework 8

1. Compute $\nabla_{\mathbf{v}}f(p)$ where $f: \mathbf{R}^{n+1} \rightarrow \mathbf{R}$, $\mathbf{v} \in \mathbf{R}_p^{n+1}$ and $p \in \mathbf{R}^{n+1}$ are given by
 - (a) $n = 1$, $\mathbf{v} = (1, 0, 2, 1)$ and $f(x_1, x_2) = 2x_1^2 + 3x_2^2$,
 - (b) $n = 1$, $\mathbf{v} = (1, 1, \cos \theta, \sin \theta)$ and $f(x_1, x_2) = x_1^2 - x_2^2$,
 - (c) $n = 2$, $\mathbf{v} = (1, 1, 1, a, b, c)$ and $f(x_1, x_2, x_3) = x_1x_2x_3^2$, and
 - (d) $n \in \mathbf{N}$, $\mathbf{v} = (p, v)$ and $f(q) = q \cdot q$.
2. Let U be an open set in \mathbf{R}^{n+1} and let $f: U \rightarrow \mathbf{R}$ be a smooth function. Show that if $\mathbf{e}_i = (0, \dots, 1, \dots, 0) \in \mathbf{R}^{n+1}$ has a 1 in the i -th coordinate and 0 in all others, and $\mathbf{e}_i = (p, \mathbf{e}_i) \in \mathbf{R}_p^{n+1}$ for $i = 1, 2, \dots, n+1$, then $\nabla_{\mathbf{e}_i}f(p) = \partial_i f(p)$.
3. Compute $\nabla_{\mathbf{v}}\mathbf{X}(p)$ where $\mathbf{v} \in \mathbf{R}_p^{n+1}$, $p \in \mathbf{R}^{n+1}$ and \mathbf{X} are given by
 - (a) $n = 1$, $\mathbf{v} = (1, 0, 0, 1)$ and $\mathbf{X}(x_1, x_2) = (x_1, x_2, x_1x_2, x_2^2)$,
 - (b) $n = 1$, $\mathbf{v} = (\cos \theta, \sin \theta, -\sin \theta, \cos \theta)$ and $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1)$, and
 - (c) $n \in \mathbf{N}$, $\mathbf{v} = (0, 0, \dots, 0, 1, 1, \dots, 1)$ and $\mathbf{X}(q) = (q, 2q)$.
4. Let $p \in S$, $\mathbf{v} \in S_p$, and \mathbf{X} and \mathbf{Y} be smooth vector fields on S . Verify that differentiation of vector fields has the following properties.
 - (a) $\nabla_{\mathbf{v}}(\mathbf{X} + \mathbf{Y})(p) = \nabla_{\mathbf{v}}\mathbf{X}(p) + \nabla_{\mathbf{v}}\mathbf{Y}(p)$,
 - (b) $\nabla_{\mathbf{v}}(f\mathbf{X})(p) = (\nabla_{\mathbf{v}}f(p))\mathbf{X}(p) + f(p)(\nabla_{\mathbf{v}}\mathbf{X}(p))$, and
 - (c) $\nabla_{\mathbf{v}}(\mathbf{X} \cdot \mathbf{Y})(p) = (\nabla_{\mathbf{v}}\mathbf{X}(p)) \cdot \mathbf{Y}(p) + \mathbf{X}(p) \cdot (\nabla_{\mathbf{v}}\mathbf{Y}(p))$.
5. A smooth tangent vector field \mathbf{X} on an n -surface S is said to be a *geodesic vector field*, or *geodesic flow*, if all integral curves of \mathbf{X} are geodesics of S .
 - (a) Show that a smooth tangent vector field \mathbf{X} on S is a geodesic field if and only if $\nabla_{\mathbf{X}(p)}\mathbf{X}(p) \perp S_p$ for all $p \in S$.
 - (b) Describe a geodesic flow on a 2-surface of revolution in \mathbf{R}^3 .
6. Choose an orientation and compute the Weingarten map for
 - (a) the hyperplane $\{(x_1, x_2, \dots, x_{n+1}) \mid a_1x_1 + a_2x_2 + \dots + a_{n+1}x_{n+1} = b\} \subset \mathbf{R}^{n+1}$ (with $(a_1, a_2, \dots, a_{n+1}) \in \mathbf{R}^{n+1} \setminus \mathbf{0}$ and $b \in \mathbf{R}$), and
 - (b) the cylinder $\{(x_1, x_2, x_3) \mid x_2^2 + x_3^2 = a^2\} \in \mathbf{R}^3$ ($a \neq 0$).
7. Let $S = f^{-1}(c)$ be an n -surface in \mathbf{R}^{n+1} oriented by $\nabla f / \|\nabla f\|$. Suppose that $p \in S$ is such that $\nabla f(p) / \|\nabla f\| = \mathbf{e}_{n+1}$, where \mathbf{e}_i are as in Question 2 ($i = 1, 2, \dots, n+1$). Show that the matrix that corresponds to L_p with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ for S_p is

$$\left(-\frac{1}{\|\nabla f(p)\|} \partial_{i,j}^2 f(p) \right)_{i,j}.$$

8. Let S be an n -surface in \mathbf{R}^{n+1} oriented by the unit normal vector field \mathbf{n} . Suppose that \mathbf{X} and \mathbf{Y} are smooth tangent vector fields on S .
 - (a) Show that

$$\nabla_{\mathbf{X}(p)}\mathbf{Y}(p) \cdot \mathbf{n}(p) = \nabla_{\mathbf{Y}(p)}\mathbf{X}(p) \cdot \mathbf{n}(p)$$
 for all $p \in S$. [Hint: Show that both sides are equal to $L_p(\mathbf{X}(p)) \cdot \mathbf{Y}(p)$.]
 - (b) Conclude that the vector field $[\mathbf{X}, \mathbf{Y}]$ defined on S by $[\mathbf{X}, \mathbf{Y}](p) = \nabla_{\mathbf{X}(p)}\mathbf{Y}(p) - \nabla_{\mathbf{Y}(p)}\mathbf{X}(p)$ is everywhere tangent to S . ($[\mathbf{X}, \mathbf{Y}]$ is called the *Lie bracket* of the vector fields \mathbf{X} and \mathbf{Y} .)