Geometry
Second Semester 2010/11
Homework 6

1. Describe the spherical image when $n = 1$ and when $n = 2$ of the n-surface $f^{-1}(0)$ oriented by $\nabla f/\|\nabla f\|$, where f is defined by
   (a) $f(x_1, x_2, \dots, x_{n+1}) = x_2^2 + \cdots + x_{n+1}^2 - 1$ (the cylinder)
   (b) $f(x_1, x_2, \dots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_{n+1}^2 - r^2$ for $r > 0$ fixed (the sphere)
   (c) $f(x_1, x_2, \dots, x_{n+1}) = -x_1 + x_2^2 + \cdots + x_{n+1}^2$ (the paraboloid)
   (d) $f(x_1, x_2, \dots, x_{n+1}) = -(x_1^2/a^2) + x_2^2 + \cdots + x_{n+1}^2 - 1$ for $a > 0$ fixed (the 1-sheeted hyperboloid)

2. What happens to the spherical image of $f^{-1}(0)$ in part (d) of Question 1 as $a \to \infty$? What happens as $a \to 0$?

3. Show that the spherical image of the n-surface with orientation $\mathbf{n}$ is the reflection through the origin of the spherical image of the same n-surface with orientation $-\mathbf{n}$.

4. Let $a = (a_1, a_2, \dots, a_{n+1}) \in \mathbf{R}^{n+1}$ be non-zero. Show that the spherical image of an n-surface S is contained in the n-plane $\{(x_1, x_2, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid a_1x_1 + a_2x_2 + \cdots + a_{n+1}x_{n+1} = 0\}$ if and only if for every $p \in S$ there is an open interval I containing 0 such that $p + ta \in S$ for all $t \in I$. [Hint: For the 'only if' part, apply Corollary 5.3 to the constant vector field $\mathbf{X}(q) = (q, a)$.]

5. Let $S = f^{-1}(c)$ for a smooth function $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$ such that $\nabla f(p) \neq \mathbf{0}$ for all $p \in S$. Suppose that $\alpha\colon \mathbf{R} \to \mathbf{R}^{n+1}$ is a parametrised curve such that $\nabla f(\alpha(t)) \cdot \dot\alpha(t) \neq 0$ for all $t \in \mathbf{R}$ such that $\alpha(t) \in S$ (when such a condition holds, we say α is *nowhere tangent* to S).
   (a) Show that at each pair of consecutive crossings of S by α, the direction of the orientation $\nabla f/\|\nabla f\|$ on S reverses relative to the direction of α. That is, show that if $\alpha(t_1) \in S$ and $\alpha(t_2) \in S$ where $t_1 < t_2$ and $\alpha(t) \notin S$ for $t_1 < t < t_2$, then $\nabla f(\alpha(t_1)) \cdot \dot\alpha(t_1) < 0$ if and only if $\nabla f(\alpha(t_2)) \cdot \dot\alpha(t_2) > 0$.
   (b) Show that if S is compact and $\lim_{t\to-\infty} \|\alpha(t)\| = \lim_{t\to\infty} \|\alpha(t)\| = \infty$ then α crosses S an even number of times.

6. Let S be a compact n-surface in $\mathbf{R}^{n+1}$. A point $p \in \mathbf{R}^{n+1} \setminus S$ is said to be *outside* S if there exists a continuous map $\alpha\colon [0, \infty) \to \mathbf{R}^{n+1} \setminus S$ such that $\alpha(0) = p$ and $\lim_{t\to\infty} \|\alpha(t)\| = \infty$. Let $\mathscr{O}(S)$ denote the set points outside S.
   (a) Show that if $\beta\colon [a, b] \to \mathbf{R}^{n+1} \setminus S$ is continuous and $\beta(a) \in \mathscr{O}(S)$ then $\beta(t) \in \mathscr{O}(S)$ for all $t \in [a, b]$.
   (b) Show that $\mathscr{O}(S)$ is a connected open subset of $\mathbf{R}^{n+1}$.