Geometry
Second Semester 2010/11
Homework 5

1. Show that the unit sphere $\mathbf{S}^n$ defined in Example 4.2 is connected if $n \geq 1$.

2. Show that if S is a connected n-surface in $\mathbf{R}^{n+1}$ and $g\colon S \to \mathbf{R}$ is smooth and takes on only the values 1 and -1, then g is constant. [Hint: Let $p \in S$. For $q \in S$, let $\alpha\colon [a,b] \to S$ be a continuous function such that $\alpha(a) = p$ and $\alpha(b) = q$. Apply the Intermediate Value Theorem to the composition $\cancel{f \circ \alpha}$ $g \circ \alpha$.]

3. Find an example that demonstrates that Theorem 5.5 fails for S which are not connected.

4. Show that two orientations of the n-sphere $\mathbf{S}^n(r)$ of radius r, defined to be $\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = r^2\}$, are given by $\mathbf{n}_1(p) = (p, p/r)$ and $\mathbf{n}_2(p) = (p, -p/r)$.

5. Consider the n-surface $S = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_{n+1} = 0\} = \mathbf{R}^n \times \{0\}$ (which can be identified with $\mathbf{R}^n$ in the obvious manner). Let $\mathbf{n}$ be the orientation on S defined by $\mathbf{n}(p) = (p, 0, 0, \ldots, 0, 1)$ for each $p \in \mathbf{R}^{n+1}$. Show that, with respect to this orientation,

   (a) when $n = 1$, the positive tangent direction at $p \in \mathbf{R}$ is the direction $(p, 1, 0)$,

   (b) when $n = 2$, the positive θ-rotation in $\mathbf{R}^2_p$ is a anticlockwise rotation through the angle θ for each $p \in \mathbf{R}^2$, and

   (c) the ordered orthogonal basis $\{(p, 1, 0, 0, 0), (p, 0, 1, 0, 0), (p, 0, 0, 1, 0), \cancel{(p, 0, 0, 0, 1)}\}$ for $\mathbf{R}^3_p$ (with $p \in \mathbf{R}^3$) is right-handed. [There should only be three elements in this basis, the fourth was a mistake in the first version.]

6. Let C be an oriented plane curve and let $\mathbf{v}$ be a non-zero vector tangent to the curve C at $p \in C$. Show that the basis $\{\mathbf{v}\}$ for C_p is consistent with the orientation on C if and only if the positive tangent direction at p is $\mathbf{v}/\|\mathbf{v}\|$. [Hint: Let θ denote the angle measured anticlockwise from $(p, 1, 0)$ to the orientation $\mathbf{n}(p)$, so that $\mathbf{n}(p) = (p, \cos\theta, \sin\theta)$. Express both $\mathbf{v}$ and the positive tangent direction at p in terms of θ.]

7. Recall that the cross-product $\mathbf{v} \times \mathbf{w}$ of two vectors $\mathbf{v} = (p, v_1, v_2, v_3)$ and $\mathbf{w} = (p, w_1, w_2, w_3)$ in $\mathbf{R}^3_p$ is defined by

$$\mathbf{v} \times \mathbf{w} = (p, v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).$$

   (a) Show that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, and that $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, where $\theta = \arccos(\mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\|\mathbf{w}\|)$ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

   (b) Show that if $\mathbf{u} = (p, u_1, u_2, u_3)$, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

   (c) Show that the only vector $\mathbf{x}$ such that $\mathbf{u} \cdot \mathbf{x}$ is equal to the determinant in (b) for all $\mathbf{u} \in \mathbf{R}^3_p$ is $\mathbf{x} = \mathbf{v} \times \mathbf{w}$.

8. Let S be an oriented 3-surface in $\mathbf{R}^4$ and let $p \in S$. This question generalises the notion of the cross product in $\mathbf{R}^3_p$ to the tangent space S_p of a general oriented 3-surfaces S in $\mathbf{R}^4$. Recall Question 5 shows us how $\mathbf{R}^3$ can be thought of as an 3-surface in $\mathbf{R}^4$.

   (a) Show that given vectors $\mathbf{v} = (p, v)$ and $\mathbf{w} = (p, w)$ in S_p, there is a unique vector, which we denote as $\mathbf{v} \times \mathbf{w}$, belonging to S_p such that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{pmatrix} u \\ v \\ w \\ N(p) \end{pmatrix}$$

   for all $\mathbf{u} = (p, u) \in S_p$, where $\mathbf{n} = (\cdot, N(\cdot))$ is the orientation of S. The vector $\mathbf{v} \times \mathbf{w}$ is the *cross product* of $\mathbf{v}$ and $\mathbf{w}$ in S_p.

(b) Check that the cross product in S_p has the following properties. For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in S_p$ and $c \in \mathbf{R}$ we have:

i. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$;

ii. $\mathbf{v} \times (\mathbf{w} + \mathbf{u}) = \mathbf{v} \times \mathbf{w} + \mathbf{v} \times \mathbf{u}$;

iii. $(c\mathbf{v}) \times \mathbf{w} = c(\mathbf{v} \times \mathbf{w})$;

iv. $\mathbf{v} \times (c\mathbf{w}) = c(\mathbf{v} \times \mathbf{w})$;

v. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$;

vi. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$;

vii. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$;

viii. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly ~~independent~~ dependent; and

ix. An ordered orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for S_p is right-handed if and only if $\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2) > 0$.