Geometry
Second Semester 2010/11
Homework 3

In Chapter 3 we have shown that at each regular point p on the level set $f^{-1}(c)$ of a smooth function f there is a well-defined vector space consisting of all velocity vectors at p of parametrised curves in $f^{-1}(c)$ passing through p. We call this vector space the *tangent space* at p and it is precisely $(\nabla f(p))^{\perp}$. It is an n-dimensional subspace of the vector space $\mathbf{R}_p^{n+1}$.

In Questions 1-2 we explore what can happen when p is not a regular point.

1. Sketch the level sets $f^{-1}(-1)$, $f^{-1}(0)$ and $f^{-1}(1)$ for $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \ldots + x_{n+1}^2$, and $n = 1$ and 2. Which points p of the level sets fail to have tangent spaces equal to $(\nabla f(p))^{\perp}$?

2. Find an examples of the following. In each case demonstrate that they have the required properties.

   (a) A function f, $c \in \mathbf{R}$ and $p \in f^{-1}(c)$ such that the set of vectors tangent to $f^{-1}(c)$ at p is not a vector subspace of $\mathbf{R}_p^{n+1}$.

   (b) A function f, $c \in \mathbf{R}$ and $p \in f^{-1}(c)$ such that the set of vectors tangent to $f^{-1}(c)$ at p is equal to $\mathbf{R}_p^{n+1}$.

3. For each f below sketch the level set $f^{-1}(0)$ and typical values of $\nabla f(p)$ of the vector field ∇f for $p \in f^{-1}(0)$.

   (a) $f(x_1, x_2) = x_1^2 + x_2^2 - 1$

   (b) $f(x_1, x_2) = x_1^2 - x_2^2 - 1$

   (c) $f(x_1, x_2) = x_1^2 - x_2^2$

   (d) $f(x_1, x_2) = x_1 - x_2^2$

4. Let $f\colon U \to \mathbf{R}$ be a smooth function, where $U \subset \mathbf{R}^{n+1}$ is an open set, and let $\alpha\colon I \to U$ be a parametrised curve. Show that $f \circ \alpha$ is constant if and only if α is everywhere orthogonal to the gradient of f (that is, if and only if $\dot{\alpha}(t) \perp \nabla f(\alpha(t))$ for all $t \in I$).

5. Let $f\colon U \to \mathbf{R}$ be a smooth function, where $U \subset \mathbf{R}^{n+1}$ is an open set, and let $\alpha\colon I \to U$ be an integral curve of ∇f.

   (a) Show that $(f \circ \alpha)'(t) = \|\nabla f(\alpha(t))\|^2$ for all $t \in I$.

   (b) Show that for each $t_0 \in I$, the function f is increasing faster along α at $\alpha(t_0)$ than along any other curve passing through $\alpha(t_0)$ with the same speed. (That is, show that if $\beta\colon \widetilde{I} \to U$ is such that $\beta(s_0) = \alpha(t_0)$ for some $s_0 \in \widetilde{I}$ and $\|\dot{\beta}(s_0)\| = \|\dot{\alpha}(t_0)\|$, then $(f \circ \alpha)'(t_0) \geq (f \circ \beta)'(s_0)$.)