Geometry
Second Semester 2010/11
Homework 2

1. Sketch the following vector fields $\mathbf{X} = (\cdot, X(\cdot))$ on $\mathbf{R}^2$ where X is defined as follows.
   (a) $X(p) = (0, 1)$
   (b) $X(p) = -p$
   (c) $X(x_1, x_2) = (x_2, -x_1)$
   (d) $X(x_1, x_2) = (x_2, x_1)$
   (e) $X(x_1, x_2) = (-2x_2, \frac{1}{2}x_1)$

2. Find and sketch the gradient of $f\colon \mathbf{R}^2 \to \mathbf{R}$ for each of the following examples.
   (a) $f(x_1, x_2) = x_1 + x_2$
   (b) $f(x_1, x_2) = x_1^2 + x_2^2$
   (c) $f(x_1, x_2) = x_1 - x_2^2$
   (d) $f(x_1, x_2) = (x_1^2 - x_2^2)/4$

3. The *divergence* of a smooth vector field $\mathbf{X}$ on $U \subset \mathbf{R}^{n+1}$, where
$$\mathbf{X}(p) = (p, X_1(p), X_2(p), \ldots, X_{n+1}(p)) \quad \text{for} \quad p \in U$$
is a function $\mathrm{div}(\mathbf{X})\colon U \to \mathbf{R}$ defined by $\mathrm{div}(\mathbf{X}) = \sum_{i=1}^{n+1} \partial_i X_i$. Find the divergence of each of the vector fields in Exercises 1 and 2.

4. Find the integral curve through $(1, 1)$ of each of the vector fields in Exercise 1.

5. Find the integral curve through the general point (a, b) of each of the vector fields in Exercise 1.

6. A smooth vector field $\mathbf{X}$ on an open set $U \in \mathbf{R}^{n+1}$ is said to be *complete* if for each $p \in U$ the maximal integral curve of $\mathbf{X}$ through p has domain equal to $\mathbf{R}$. Determine which of the following vector fields are complete.
   (a) $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1, 0)$, $U = \mathbf{R}^2$
   (b) $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1, 0)$, $U = \mathbf{R}^2 \setminus \{(0, 0)\}$
   (c) $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1)$, $U = \mathbf{R}^2 \setminus \{(0, 0)\}$
   (d) $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1 + x_1^2, 0)$, $U = \mathbf{R}^2$

7. Recall that a subset U of $\mathbf{R}^N$ is said to be open if for each $q \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(q) \subset U$. Here $B_\varepsilon(q) := \{p \in \mathbf{R}^n \mid \|p - q\| < \varepsilon\}$ is a ball of radius ε with centre q. Prove that any union of open sets is itself an open set. That is, let Γ be any given index set and suppose $\{S_\gamma\}_{\gamma \in \Gamma}$ is a collection of open sets. Show that
$$\bigcup_{\gamma \in \Gamma} S_\gamma$$
is an open set.

8. Suppose that U is an open set in $\mathbf{R}^{n+1}$ and let $p \in U$. Given a smooth vector field $\mathbf{X}$ on U, let $\alpha\colon I \to U$ be the maximal integral curve of $\mathbf{X}$ through p. Show that if $\beta\colon \widetilde{I} \to U$ is any integral curve of $\mathbf{X}$ with $\beta(t_0) = p$ for some $t_0 \in \widetilde{I}$, then $\beta(t) = \alpha(t - t_0)$ for all $t \in \widetilde{I}$.

9. Let U be an open set in $\mathbf{R}^{n+1}$, $\mathbf{X}$ be a smooth vector field on U and $\alpha\colon I \to U$ the maximal integral curve of $\mathbf{X}$ through a point. Suppose that $\alpha(0) = \alpha(t_0)$ for some $t_0 \in I \setminus \{0\}$. Show that $I = \mathbf{R}$ and $\alpha(t + t_0) = \alpha(t)$ for all $t \in \mathbf{R}$.

10. Consider the vector field $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1, 0)$ on $\mathbf{R}^2$. For $t \in \mathbf{R}$ and $p \in \mathbf{R}^2$, let $\varphi_t(p) = \alpha_p(t)$, where α_p is the maximal integral curve of $\mathbf{X}$ through p.

(a) Show that, for each t, φ_t is a one-to-one transformation from $\mathbf{R}^2$ onto itself. (That is, show that $\varphi_t \colon \mathbf{R}^2 \to \mathbf{R}^2$ is injective and surjective.) Geometrically, what does this transformation do?

(b) Show that φ_0 is the identity, $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$ for all $t_1, t_2 \in \mathbf{R}$ and $\varphi_{-t} = \varphi_t^{-1}$ for all $t \in \mathbf{R}$. (This shows that $t \mapsto \varphi_t$ is a homeomorphism from the additive group of real numbers into the group of one-to-one transformations of the plane.)

11. Repeat Exercise 10 for the following vector fields.

(a) $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1)$

(b) $\mathbf{X}(x_1, x_2) = (x_1, x_2, x_1, x_2)$

(c) $\mathbf{X}(x_1, x_2) = (x_1, x_2, x_2, x_1)$

12. Let $\mathbf{X}$ be any smooth vector field on an open set $U \subset \mathbf{R}^{n+1}$. Let $\varphi_t(p) = \alpha_p(t)$, where α_p is the maximal integral curve of $\mathbf{X}$ through p. Use the uniqueness of integral curves to show that $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$ and $\varphi_{-t} = \varphi_t^{-1}$ for all t, t_1, t_2 for which all the terms are defined. (φ_t is called the *local 1-parameter group* associated to $\mathbf{X}$.)