(1) Let $f\colon U \to \mathbf{R}$ be a smooth function defined on an open set $U \subseteq \mathbf{R}^{n+1}$.

(a) Define what it means for a vector $\mathbf{v}$ based at p to be tangent to the set $f^{-1}(c)$ at p for $c \in \mathbf{R}$ and $p \in f^{-1}(c)$. **[3 marks]**

(b) State precisely a theorem which characterises the set of all tangent vectors to $f^{-1}(c)$ at a point $p \in f^{-1}(c)$. What condition on f must hold? **[7 marks]**

(c) Prove that the gradient $\nabla f(p)$ of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p. **[5 marks]**

(d) Find a function f, $c \in \mathbf{R}$ and $p \in f^{-1}(c)$ such that the set of vectors tangent to $f^{-1}(c)$ at p is not a vector subspace of $\mathbf{R}^{n+1}_p$. **[5 marks]**

(2) Let S be an n-surface in $\mathbf{R}^{n+1}$ with $S = f^{-1}(c)$ for some smooth function $f\colon U \to \mathbf{R}$ and $c \in \mathbf{R}$, where U is an open subset of $\mathbf{R}^{n+1}$.

(a) Define what it means for a vector field $\mathbf{X}$ on S to be a normal vector field. **[3 marks]**

(b) Define what it means to say S is connected. **[3 marks]**

(c) Give two choices of unit normal vector field for S. **[5 marks]**

(d) Suppose S is connected. Prove that there are at most two choices of unit normal vector fields. **[9 marks]**

(3) (a) Define what it means for a parametrised curve on an oriented n-surface S to be a geodesic. **[3 marks]**

(b) State precisely a theorem regarding the existence and uniqueness of maximal geodesics on an oriented n-surface S. **[3 marks]**

(c) Let $p \in \mathbf{S}^n$ and $\mathbf{v} = (p, v) \in \mathbf{S}^n_p$. Define a parametrised curve $\alpha\colon \mathbf{R} \to \mathbf{R}^{n+1}$ by

$$\alpha(t) = (\cos(at))p + (\sin(at))q,$$

where $q = v/\|\mathbf{v}\|$ and $a \in \mathbf{R}$, for all $t \in \mathbf{R}$.

(i) Show that, in fact, $\alpha\colon \mathbf{R} \to \mathbf{S}^n$.

(ii) Show that $\alpha(0) = p$. Determine the value of $a \in \mathbf{R}$ for which $\dot{\alpha}(0) = \mathbf{v}$.

(iii) For the value of a calculated in (3(c)ii), prove that α is the maximal geodesic of $\mathbf{S}^n$ passing through p with velocity $\mathbf{v}$.

(4) (a) Let S be an oriented n-surface with orientation $\mathbf{n}$. Define the Weingarten map L_p for a $p \in S$. **[5 marks]**

[Please turn over]

(b) State precisely a theorem which relates the Weingarten map $L_p(\mathbf{v})$ at $\mathbf{v} \in S_p$ and the acceleration of a parametrised curve $\alpha\colon I \to S$ with velocity $\dot{\alpha}(t_0) = \mathbf{v}$. **[7 marks]**

(c) Show that all integral curves of a smooth tangent vector field $\mathbf{X}$ on S are geodesics if and only if $\nabla_{\mathbf{X}(p)}\mathbf{X}(p) \perp S_p$ for all $p \in S$. **[8 marks]**

[End of Paper]