Geometry
Second Semester 2011/12
Homework 9

1. Let $\alpha\colon I \to C$ be a local parametrisation of the oriented curve C of the form $\alpha(t) = (x(t), y(t))$. Show that
$$(\kappa \circ \alpha)(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{(x'(t)^2 + y'(t)^2)^{\frac{3}{2}}}.$$

2. Find global parametrisations of $f^{-1}(0)$ oriented by $\nabla f / \|\nabla f\|$ in each of the following cases:

   (a) $f(x_1, x_2) = ax_1 + bx_2 - c$ with $(a, b) \neq (0, 0)$;
   (b) $f(x_1, x_2) = x_1^2/a^2 + x_2^2/b^2 - 1$ with $a, b \neq 0$;
   (c) $f(x_1, x_2) = x_2 - ax_1^2 - 1$ with $a \neq 0$; and
   (d) $f(x_1, x_2) = x_1^2 - x_2^2 - 1$ defined for $x_1 > 0$.

3. Find the curvature of each of the oriented plane curves in Exercise 2.

4. Let C be a plane curve oriented by the unit normal vector field $\mathbf{n}$. Let $\alpha\colon I \to C$ be a unit speed local parametrisation for C. For $t \in I$, let $\mathbf{T}(t) = \dot\alpha(t)$. Show that
$$\begin{cases} \dot{\mathbf{T}} = (\kappa \circ \alpha)(\mathbf{n} \circ \alpha), & \text{and} \\ (\mathbf{n} \circ \alpha)^{\cdot} = -(\kappa \circ \alpha)\mathbf{T}. \end{cases}$$
These formulae are called the *Frenet formulae* for a plane curve.

5. Let $S = f^{-1}(c)$ be an n-surface in $\mathbf{R}^{n+1}$ oriented by $\nabla f / \|\nabla f\|$. For $\mathbf{v} = (p, v_1, v_2, \ldots, v_{n+1}) \in S_p$, with $p \in S$, we define the *second fundamental form* of S at p on $\mathbf{v}$ to be $\mathscr{S}_p(\mathbf{v}) := L_p(\mathbf{v}) \cdot \mathbf{v}$. Show that
$$\mathscr{S}_p(\mathbf{v}) = \frac{-1}{\|\nabla f\|} \sum_{i,j=1}^{n+1} \partial^2_{i,j} f(p) v_i v_j.$$
(When $\|\mathbf{v}\| = 1$, then $k(\mathbf{v}) = \mathscr{S}_p(\mathbf{v})$, so this formula provides a useful way to compute normal curvatures.)

6. Find the normal curvature $k(\mathbf{v})$ for each direction $\mathbf{v}$, the principal curvatures and principal curvature directions at the point p for the 3-surface $f^{-1}(0)$ oriented by $\nabla f / \|\nabla f\|$ in each of the following cases.

   (a) $f(x_1, x_2, x_3) = x_1^2/a^2 + x_2^2/b^2 + x_3^2/c^2 - 1$, $p = (a, 0, 0)$
   (b) $f(x_1, x_2, x_3) = x_1^2/a^2 + x_2^2/b^2 - x_3^2/c^2 - 1$, $p = (a, 0, 0)$
   (c) $f(x_1, x_2, x_3) = x_1^2 + (\sqrt{x_2^2 + x_3^2} - 2)^2 - 1$,
      i. $p = (0, 3, 0)$
      ii. $p = (0, 1, 0)$