Geometry
Second Semester 2011/12
Homework 8

1. Compute $\nabla_{\mathbf{v}} f(p)$ where $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$, $\mathbf{v} \in \mathbf{R}^{n+1}_p$ and $p \in \mathbf{R}^{n+1}$ are given by
   (a) $n = 1$, $\mathbf{v} = (1, 0, 2, 1)$ and $f(x_1, x_2) = 2x_1^2 + 3x_2^2$,
   (b) $n = 1$, $\mathbf{v} = (1, 1, \cos\theta, \sin\theta)$ and $f(x_1, x_2) = x_1^2 - x_2^2$,
   (c) $n = 2$, $\mathbf{v} = (1, 1, 1, a, b, c)$ and $f(x_1, x_2, x_3) = x_1 x_2 x_3^2$, and
   (d) $n \in \mathbf{N}$, $\mathbf{v} = (p, v)$ and $f(q) = q \cdot q$.

2. Let U be an open set in $\mathbf{R}^{n+1}$ and let $f\colon U \to \mathbf{R}$ be a smooth function. Show that if $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbf{R}^{n+1}$ has a 1 in the i-th coordinate and 0 in all others, and $\mathbf{e}_i = (p, e_i) \in \mathbf{R}^{n+1}_p$ for $i = 1, 2, \ldots, n+1$, then $\nabla_{\mathbf{e}_i} f(p) = \partial_i f(p)$.

3. Compute $\nabla_{\mathbf{v}} \mathbf{X}(p)$ where $\mathbf{v} \in \mathbf{R}^{n+1}_p$, $p \in \mathbf{R}^{n+1}$ and $\mathbf{X}$ are given by
   (a) $n = 1$, $\mathbf{v} = (1, 0, 0, 1)$ and $\mathbf{X}(x_1, x_2) = (x_1, x_2, x_1 x_2, x_2^2)$,
   (b) $n = 1$, $\mathbf{v} = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta)$ and $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1)$, and
   (c) $n \in \mathbf{N}$, $\mathbf{v} = (0, 0, \ldots, 0, 1, 1, \ldots, 1)$ and $\mathbf{X}(q) = (q, 2q)$.

4. Let $p \in S$, $\mathbf{v} \in S_p$, and $\mathbf{X}$ and $\mathbf{Y}$ be smooth vector fields on S. Verify that differentiation of vector fields has the following properties.
   (a) $\nabla_{\mathbf{v}}(\mathbf{X} + \mathbf{Y})(p) = \nabla_{\mathbf{v}} \mathbf{X}(p) + \nabla_{\mathbf{v}} \mathbf{Y}(p)$,
   (b) $\nabla_{\mathbf{v}}(f\mathbf{X})(p) = (\nabla_{\mathbf{v}} f(p))\mathbf{X}(p) + f(p)(\nabla_{\mathbf{v}} \mathbf{X}(p))$, and
   (c) $\nabla_{\mathbf{v}}(\mathbf{X} \cdot \mathbf{Y})(p) = (\nabla_{\mathbf{v}} \mathbf{X}(p)) \cdot \mathbf{Y}(p) + \mathbf{X}(p) \cdot (\nabla_{\mathbf{v}} \mathbf{Y}(p))$.

5. A smooth tangent vector field $\mathbf{X}$ on an n-surface S is said to be a *geodesic vector field*, or *geodesic flow*, if all integral curves of $\mathbf{X}$ are geodesics of S.
   (a) Show that a smooth tangent vector field $\mathbf{X}$ on S is a geodesic field if and only if $\nabla_{\mathbf{X}(p)} \mathbf{X}(p) \perp S_p$ for all $p \in S$.
   (b) Describe a geodesic flow on a 2-surface of revolution in $\mathbf{R}^3$.

6. Choose an orientation and compute the Weingarten map for
   (a) the hyperplane $\{(x_1, x_2, \ldots, x_{n+1}) \,|\, a_1 x_1 + a_2 x_2 + \cdots + a_{n+1} x_{n+1} = b\} \subset \mathbf{R}^{n+1}$ (with $(a_1, a_2, \ldots, a_{n+1}) \in \mathbf{R}^{n+1} \setminus \mathbf{0}$ and $b \in \mathbf{R}$), and
   (b) the cylinder $\{(x_1, x_2, x_3) \,|\, x_2^2 + x_3^2 = a^2\} \in \mathbf{R}^3$ ($a \neq 0$).

7. Let $S = f^{-1}(c)$ be an n-surface in $\mathbf{R}^{n+1}$ oriented by $\nabla f / \|\nabla f\|$. Suppose that $p \in S$ is such that $\nabla f(p) / \|\nabla f(p)\| = \mathbf{e}_{n+1}$, where $\mathbf{e}_i$ are as in Question 2 ($i = 1, 2, \ldots, n+1$). Show that the matrix that corresponds to L_p with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for S_p is
$$\left(-\frac{1}{\|\nabla f(p)\|} \partial^2_{i,j} f(p)\right)_{i,j}.$$

8. Let S be an n-surface in $\mathbf{R}^{n+1}$ oriented by the unit normal vector field $\mathbf{n}$. Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are smooth tangent vector fields on S.
   (a) Show that
$$\nabla_{\mathbf{X}(p)} \mathbf{Y}(p) \cdot \mathbf{n}(p) = \nabla_{\mathbf{Y}(p)} \mathbf{X}(p) \cdot \mathbf{n}(p)$$
   for all $p \in S$. [Hint: Show that both sides are equal to $L_p(\mathbf{X}(p)) \cdot \mathbf{Y}(p)$.]
   (b) Conclude that the vector field $[\mathbf{X}, \mathbf{Y}]$ defined on S by $[\mathbf{X}, \mathbf{Y}](p) = \nabla_{\mathbf{X}(p)} \mathbf{Y}(p) - \nabla_{\mathbf{Y}(p)} \mathbf{X}(p)$ is everywhere tangent to S. ($[\mathbf{X}, \mathbf{Y}]$ is called the *Lie bracket* of the vector fields $\mathbf{X}$ and $\mathbf{Y}$.)