Geometry
Second Semester 2011/12
Homework 7

1. Find the velocity, acceleration and speed of each of the following parametrised curves $\alpha\colon \mathbf{R} \to \mathbf{R}^{n+1}$.
   (a) $n = 1$ and $\alpha(t) = (t, t^2)$,
   (b) $n = 1$ and $\alpha(t) = (\cos t, \sin t)$,
   (c) $n = 1$ and $\alpha(t) = (\cos 3t, \sin 3t)$,
   (d) $n = 2$ and $\alpha(t) = (\cos t, \sin t, t)$, and
   (e) $n = 3$ and $\alpha(t) = (\cos t, \sin t, 2\cos t, 2\sin t)$.
2. Show that if $\alpha\colon I \to \mathbf{R}^{n+1}$ is a parametrised curve with constant speed, then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all $t \in I$.
3. Let $\alpha\colon I \to \mathbf{R}^{n+1}$ be a parametrised curve such that $\dot{\alpha}(t) \neq \mathbf{0}$ for all $t \in I$. Show that there exists a unit speed reparametrisation β of α. That is, show that there exists an interval J and a smooth surjective function $h\colon J \to I$ such that $h' > 0$ and $\beta = \alpha \circ h$ has unit speed. [Hint: Set $h = s^{-1}$, where $s(t) = \int_{t_0}^{t} \|\dot{\alpha}(\tau)\| d\tau$ for some $t_0 \in I$.]
4. Let $\mathbf{X}$ and $\mathbf{Y}$ be smooth vector fields along the parametrised curve $\alpha\colon I \to \mathbf{R}^{n+1}$ and let $f\colon I \to \mathbf{R}$ be a smooth function along α. Verify the following identities, which we claimed to hold in class:
   (a) $(\mathbf{X} + \mathbf{Y})^{\cdot} = \dot{\mathbf{X}} + \dot{\mathbf{Y}}$;
   (b) $(f\mathbf{X})^{\cdot} = f'\mathbf{X} + f\dot{\mathbf{X}}$; and
   (c) $(\mathbf{X} \cdot \mathbf{Y})' = \dot{\mathbf{X}} \cdot \mathbf{Y} + \mathbf{X} \cdot \dot{\mathbf{Y}}$.
5. Let S be an n-surface in $\mathbf{R}^{n+1}$, $p \in S$, $\mathbf{v} \in S_p$ and let $\alpha\colon I \to S$ be the maximal geodesic in S passing through p with velocity $\mathbf{v}$. Show that the maximal geodesic β in S with $\beta(0) = p$ and $\dot{\beta}(0) = c\mathbf{v}$, with $c \in \mathbf{R}$, is give by the formula $\beta(t) = \alpha(ct)$ for all t belonging to some interval $\widetilde{I}$.
6. Let $\alpha\colon I \to S$ be a geodesic on an n-surface S and let $\beta = \alpha \circ h$, where $h\colon \widetilde{I} \to I$ is a smooth surjective function, be a reparametrisation of α. Show that β is a geodesic on S if and only if $h(t) = at + b$ for some $a, b \in \mathbf{R}$ and all $t \in \widetilde{I}$.
7. An n-surface in $\mathbf{R}^{n+1}$ is said to be *geodesically complete* if every maximal geodesic on S has domain $\mathbf{R}$. Which of the following n-surfaces are geodesically complete?
   (a) The n-sphere $\mathbf{S}^n \subset \mathbf{R}^{n+1}$ (see Example 4.2).
   (b) $f^{-1}(1) \subset \mathbf{R}^{n+1}$ where $f\colon U \to \mathbf{R}$ has domain $U = \{(x_1, x_2, \ldots, x_{n+1}) \,|\, x_{n+1} < 1\}$ and is given by $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 + x_2^2 + \cdots + x_{n+1}^2$ for each $(x_1, x_2, \ldots, x_{n+1}) \in U$. This is the n-sphere $\mathbf{S}^n \setminus \{(0, \ldots, 0, 1)\}$ with the north pole deleted.
   (c) $f^{-1}(0) \subset \mathbf{R}^3$ where $f\colon U \to \mathbf{R}$ has domain $U = \{(x_1, x_2, x_3) \,|\, x_3 > 0\}$ and is given by $f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$ for each $(x_1, x_2, \ldots, x_{n+1}) \in U$. This is a cone with the vertex deleted.
   (d) The 2-surface S in $\mathbf{R}^3$ which is the cylinder over the plane curve $\mathbf{S}^1$ (see Examples 4.2 and 4.5).