

Geometry
Second Semester 2011/12
 Homework 5

1. Show that the unit sphere \mathbf{S}^n defined in Example 4.2 is connected if $n \geq 1$.
2. Show that if S is a connected n -surface in \mathbf{R}^{n+1} and $g: S \rightarrow \mathbf{R}$ is smooth and takes on only the values 1 and -1 , then g is constant. [Hint: Let $p \in S$. For $q \in S$, let $\alpha: [a, b] \rightarrow S$ be a continuous function such that $\alpha(a) = p$ and $\alpha(b) = q$. Apply the Intermediate Value Theorem to the composition $g \circ \alpha$.]
3. Find an example that demonstrates that Theorem 5.5 fails for S which are not connected.
4. Show that two orientations of the n -sphere $\mathbf{S}^n(r)$ of radius r , defined to be $\{(x_1, x_2, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = r^2\}$, are given by $\mathbf{n}_1(p) = (p, p/r)$ and $\mathbf{n}_2(p) = (p, -p/r)$.
5. Consider the n -surface $S = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_{n+1} = 0\} = \mathbf{R}^n \times \{0\}$ (which can be identified with \mathbf{R}^n in the obvious manner). Let \mathbf{n} be the orientation on S defined by $\mathbf{n}(p) = (p, 0, 0, \dots, 0, 1)$ for each $p \in \mathbf{R}^{n+1}$. Show that, with respect to this orientation,
 - (a) when $n = 1$, the positive tangent direction at $p \in \mathbf{R}$ is the direction $(p, 1, 0)$,
 - (b) when $n = 2$, the positive θ -rotation in \mathbf{R}_p^2 is a anticlockwise rotation through the angle θ for each $p \in \mathbf{R}^2$, and
 - (c) the ordered orthogonal basis $\{(p, 1, 0, 0, 0), (p, 0, 1, 0, 0), (p, 0, 0, 1, 0)\}$ for \mathbf{R}_p^3 (with $p \in \mathbf{R}^3$) is right-handed.
6. Let C be an oriented plane curve and let \mathbf{v} be a non-zero vector tangent to the curve C at $p \in C$. Show that the basis $\{\mathbf{v}\}$ for C_p is consistent with the orientation on C if and only if the positive tangent direction at p is $\mathbf{v}/\|\mathbf{v}\|$. [Hint: Let θ denote the angle measured anticlockwise from $(p, 1, 0)$ to the orientation $\mathbf{n}(p)$, so that $\mathbf{n}(p) = (p, \cos \theta, \sin \theta)$. Express both \mathbf{v} and the positive tangent direction at p in terms of θ .]
7. Recall that the cross-product $\mathbf{v} \times \mathbf{w}$ of two vectors $\mathbf{v} = (p, v_1, v_2, v_3)$ and $\mathbf{w} = (p, w_1, w_2, w_3)$ in \mathbf{R}_p^3 is defined by

$$\mathbf{v} \times \mathbf{w} = (p, v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1).$$

- (a) Show that $\mathbf{v} \times \mathbf{w}$ is orthogonal to both \mathbf{v} and \mathbf{w} , and that $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin \theta$, where $\theta = \arccos(\mathbf{v} \cdot \mathbf{w}/\|\mathbf{v}\|\|\mathbf{w}\|)$ is the angle between \mathbf{v} and \mathbf{w} .
- (b) Show that if $\mathbf{u} = (p, u_1, u_2, u_3)$, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

- (c) Show that the only vector \mathbf{x} such that $\mathbf{u} \cdot \mathbf{x}$ is equal to the determinant in (b) for all $\mathbf{u} \in \mathbf{R}_p^3$ is $\mathbf{x} = \mathbf{v} \times \mathbf{w}$.
8. Let S be an oriented 3-surface in \mathbf{R}^4 and let $p \in S$. This question generalises the notion of the cross product in \mathbf{R}_p^3 to the tangent space S_p of a general oriented 3-surfaces S in \mathbf{R}^4 . Recall Question 5 shows us how \mathbf{R}^3 can be thought of as an 3-surface in \mathbf{R}^4 .
 - (a) Show that given vectors $\mathbf{v} = (p, v)$ and $\mathbf{w} = (p, w)$ in S_p , there is a unique vector, which we denote as $\mathbf{v} \times \mathbf{w}$, belonging to S_p such that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{pmatrix} u \\ v \\ w \\ N(p) \end{pmatrix}$$

for all $\mathbf{u} = (p, u) \in S_p$, where $\mathbf{n} = (\cdot, N(\cdot))$ is the orientation of S . The vector $\mathbf{v} \times \mathbf{w}$ is the *cross product* of \mathbf{v} and \mathbf{w} in S_p .

(b) Check that the cross product in S_p has the following properties. For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in S_p$ and $c \in \mathbf{R}$ we have:

i. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u};$

ii. $\mathbf{v} \times (\mathbf{w} + \mathbf{u}) = \mathbf{v} \times \mathbf{w} + \mathbf{v} \times \mathbf{u};$

iii. $(c\mathbf{v}) \times \mathbf{w} = c(\mathbf{v} \times \mathbf{w});$

iv. $\mathbf{v} \times (c\mathbf{w}) = c(\mathbf{v} \times \mathbf{w});$

v. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v};$

vi. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v});$

vii. $\mathbf{v} \times \mathbf{w}$ is orthogonal to both \mathbf{v} and \mathbf{w} ;

viii. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent; and

ix. An ordered orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for S_p is right-handed if and only if $\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2) > 0$.