

Geometry
Second Semester 2011/12
Homework 3

In Chapter 3 we have shown that at each regular point p on the level set $f^{-1}(c)$ of a smooth function f there is a well-defined vector space consisting of all velocity vectors at p of parametrised curves in $f^{-1}(c)$ passing through p . We call this vector space the *tangent space* at p and it is precisely $(\nabla f(p))^\perp$. It is an n -dimensional subspace of the vector space \mathbf{R}_p^{n+1} .

In Questions 1-2 we explore what can happen when p is not a regular point.

1. Sketch the level sets $f^{-1}(-1)$, $f^{-1}(0)$ and $f^{-1}(1)$ for $f(x_1, x_2, \dots, x_{n+1}) = x_1^2 + x_2^2 + \dots + x_{n+1}^2$, and $n = 1$ and 2. Which points p of the level sets fail to have tangent spaces equal to $(\nabla f(p))^\perp$?
2. Find an examples of the following. In each case demonstrate that they have the required properties.
 - (a) A function f , $c \in \mathbf{R}$ and $p \in f^{-1}(c)$ such that the set of vectors tangent to $f^{-1}(c)$ at p is not a vector subspace of \mathbf{R}_p^{n+1} .
 - (b) A function f , $c \in \mathbf{R}$ and $p \in f^{-1}(c)$ such that the set of vectors tangent to $f^{-1}(c)$ at p is equal to \mathbf{R}_p^{n+1} .
3. For each f below sketch the level set $f^{-1}(0)$ and typical values of $\nabla f(p)$ of the vector field ∇f for $p \in f^{-1}(0)$.
 - (a) $f(x_1, x_2) = x_1^2 + x_2^2 - 1$
 - (b) $f(x_1, x_2) = x_1^2 - x_2^2 - 1$
 - (c) $f(x_1, x_2) = x_1^2 - x_2^2$
 - (d) $f(x_1, x_2) = x_1 - x_2^2$
4. Let $f: U \rightarrow \mathbf{R}$ be a smooth function, where $U \subset \mathbf{R}^{n+1}$ is an open set, and let $\alpha: I \rightarrow U$ be a parametrised curve. Show that $f \circ \alpha$ is constant if and only if α is everywhere orthogonal to the gradient of f (that is, if and only if $\dot{\alpha}(t) \perp \nabla f(\alpha(t))$ for all $t \in I$).
5. Let $f: U \rightarrow \mathbf{R}$ be a smooth function, where $U \subset \mathbf{R}^{n+1}$ is an open set, and let $\alpha: I \rightarrow U$ be an integral curve of ∇f .
 - (a) Show that $(f \circ \alpha)'(t) = \|\nabla f(\alpha(t))\|^2$ for all $t \in I$.
 - (b) Show that for each $t_0 \in I$, the function f is increasing faster along α at $\alpha(t_0)$ than along any other curve passing through $\alpha(t_0)$ with the same speed. (That is, show that if $\beta: \tilde{I} \rightarrow U$ is such that $\beta(s_0) = \alpha(t_0)$ for some $s_0 \in \tilde{I}$ and $\|\dot{\beta}(s_0)\| = \|\dot{\alpha}(t_0)\|$, then $(f \circ \alpha)'(t_0) \geq (f \circ \beta)'(s_0)$.)