

Geometry
Second Semester 2011/12
Homework 2

1. Sketch the following vector fields $\mathbf{X} = (\cdot, X(\cdot))$ on \mathbf{R}^2 where X is defined as follows.

- (a) $X(p) = (0, 1)$
- (b) $X(p) = -p$
- (c) $X(x_1, x_2) = (x_2, -x_1)$
- (d) $X(x_1, x_2) = (x_2, x_1)$
- (e) $X(x_1, x_2) = (-2x_2, \frac{1}{2}x_1)$

2. Find and sketch the gradient of $f: \mathbf{R}^2 \rightarrow \mathbf{R}$ for each of the following examples.

- (a) $f(x_1, x_2) = x_1 + x_2$
- (b) $f(x_1, x_2) = x_1^2 + x_2^2$
- (c) $f(x_1, x_2) = x_1 - x_2^2$
- (d) $f(x_1, x_2) = (x_1^2 - x_2^2)/4$

3. The *divergence* of a smooth vector field \mathbf{X} on $U \subset \mathbf{R}^{n+1}$, where

$$\mathbf{X}(p) = (p, X_1(p), X_2(p), \dots, X_{n+1}(p)) \quad \text{for } p \in U$$

is a function $\text{div}(\mathbf{X}): U \rightarrow \mathbf{R}$ defined by $\text{div}(\mathbf{X}) = \sum_{i=1}^{n+1} \partial_i X_i$. Find the divergence of each of the vector fields in Exercises 1 and 2.

4. Find the integral curve through $(1, 1)$ of each of the vector fields in Exercise 1.

5. Find the integral curve through the general point (a, b) of each of the vector fields in Exercise 1.

6. A smooth vector field \mathbf{X} on an open set $U \subset \mathbf{R}^{n+1}$ is said to be *complete* if for each $p \in U$ the maximal integral curve of \mathbf{X} through p has domain equal to \mathbf{R} . Determine which of the following vector fields are complete.

- (a) $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1, 0)$, $U = \mathbf{R}^2$
- (b) $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1, 0)$, $U = \mathbf{R}^2 \setminus \{(0, 0)\}$
- (c) $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1)$, $U = \mathbf{R}^2 \setminus \{(0, 0)\}$
- (d) $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1 + x_1^2, 0)$, $U = \mathbf{R}^2$

7. Recall that a subset U of \mathbf{R}^N is said to be open if for each $q \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(q) \subset U$. Here $B_\varepsilon(q) := \{p \in \mathbf{R}^n \mid \|p - q\| < \varepsilon\}$ is a ball of radius ε with centre q . Prove that any union of open sets is itself an open set. That is, let Γ be any given index set and suppose $\{S_\gamma\}_{\gamma \in \Gamma}$ is a collection of open sets. Show that

$$\bigcup_{\gamma \in \Gamma} S_\gamma$$

is an open set.

8. Suppose that U is an open set in \mathbf{R}^{n+1} and let $p \in U$. Given a smooth vector field \mathbf{X} on U , let $\alpha: I \rightarrow U$ be the maximal integral curve of \mathbf{X} through p . Show that if $\beta: \tilde{I} \rightarrow U$ is any integral curve of \mathbf{X} with $\beta(t_0) = p$ for some $t_0 \in \tilde{I}$, then $\beta(t) = \alpha(t - t_0)$ for all $t \in \tilde{I}$.

9. Let U be an open set in \mathbf{R}^{n+1} , \mathbf{X} be a smooth vector field on U and $\alpha: I \rightarrow U$ the maximal integral curve of \mathbf{X} through a point. Suppose that $\alpha(0) = \alpha(t_0)$ for some $t_0 \in I \setminus \{0\}$. Show that $I = \mathbf{R}$ and $\alpha(t + t_0) = \alpha(t)$ for all $t \in \mathbf{R}$.

10. Consider the vector field $\mathbf{X}(x_1, x_2) = (x_1, x_2, 1, 0)$ on \mathbf{R}^2 . For $t \in \mathbf{R}$ and $p \in \mathbf{R}^2$, let $\varphi_t(p) = \alpha_p(t)$, where α_p is the maximal integral curve of \mathbf{X} through p .

- (a) Show that, for each t , φ_t is a one-to-one transformation from \mathbf{R}^2 onto itself. (That is, show that $\varphi_t: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is injective and surjective.) Geometrically, what does this transformation do?
- (b) Show that φ_0 is the identity, $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$ for all $t_1, t_2 \in \mathbf{R}$ and $\varphi_{-t} = \varphi_t^{-1}$ for all $t \in \mathbf{R}$. (This shows that $t \mapsto \varphi_t$ is a homeomorphism from the additive group of real numbers into the group of one-to-one transformations of the plane.)

11. Repeat Exercise 10 for the following vector fields.

(a) $\mathbf{X}(x_1, x_2) = (x_1, x_2, -x_2, x_1)$

(b) $\mathbf{X}(x_1, x_2) = (x_1, x_2, x_1, x_2)$

(c) $\mathbf{X}(x_1, x_2) = (x_1, x_2, x_2, x_1)$

12. Let \mathbf{X} be any smooth vector field on an open set $U \subset \mathbf{R}^{n+1}$. Let $\varphi_t(p) = \alpha_p(t)$, where α_p is the maximal integral curve of \mathbf{X} through p . Use the uniqueness of integral curves to show that $\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$ and $\varphi_{-t} = \varphi_t^{-1}$ for all t, t_1, t_2 for which all the terms are defined. (φ_t is called the *local 1-parameter group* associated to \mathbf{X} .)