Geometry
Second Semester 2011/12
Homework 1

1. Find the local minima and local maxima of the following functions. Justify the answers you obtain in each case.

   (a) $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3 - 2x^2 + x - 3$ for all $x \in \mathbf{R}$.

   (b) $g: (-3, 3] \to \mathbf{R}$ defined by $g(x) = x^3 - 2x^2 + x - 3$ for all $x \in (-3, 3]$.

   (c) $h: \mathbf{R}^2 \to \mathbf{R}$ defined by $h(x_1, x_2) = x_1^2 - 4x_1 + 4 + x_2^2$ for all $(x_1, x_2) \in \mathbf{R}^2$.

   (d) $k: \mathbf{R}^2 \to \mathbf{R}$ defined by $k(x_1, x_2) = x_1^2 - x_2^2$.

2. Let A be an $n \times n$ matrix with entries $a_{ij} \in \mathbf{R}$ in the ij-th position. Show that if

$$\mathbf{w}A\mathbf{v} = 0$$

   for all column vectors $\mathbf{v}$ and row vectors $\mathbf{w}$ (all of dimension n), then $a_{ij} = 0$ for all $i, j = 1, \ldots, n$.

3. Prove the chain rule for real-valued functions: Let $f: (a, b) \to \mathbf{R}$ be differentiable at $x \in (a, b)$ and such that $f(x) \in (c, d)$. Also let $G: (c, d) \to \mathbf{R}$ be differentiable at $f(x)$. Prove that $G \circ f$ is differentiable at x and

$$(G \circ f)'(x) = G'(f(x))f'(x).$$

4. Sketch typical level sets and the graph of the function $f: \mathbf{R}^2 \to \mathbf{R}$ when defined as follows.

   (a) $f(x_1, x_2) = x_1$ for all $(x_1, x_2) \in \mathbf{R}^2$.

   (b) $f(x_1, x_2) = x_1 - x_2$ for all $(x_1, x_2) \in \mathbf{R}^2$.

   (c) $f(x_1, x_2) = x_1^2 - x_2^2$ for all $(x_1, x_2) \in \mathbf{R}^2$.

5. Sketch the level sets $f^{-1}(c)$ for $n = 0, 1$ and 2 of the function $f: \mathbf{R}^{n+1} \to \mathbf{R}$ when defined as follows at the heights indicated.

   (a) $f(x_1, x_2, \ldots, x_{n+1}) = x_{n+1}$ for all $(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$; $c = -1, 0, 1, 2$.

   (b) $f(x_1, x_2, \ldots, x_{n+1}) = 0x_1^2 + x_2^2 + \ldots + x_{n+1}^2$ for all $(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$; $c = 0, 1, 4$.

   (c) $f(x_1, x_2, \ldots, x_{n+1}) = x_1 - x_2^2 + \ldots - x_{n+1}^2$ for all $(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$; $c = -1, 0, 1, 2$.

   (d) $f(x_1, x_2, \ldots, x_{n+1}) = x_1^2 - x_2^2 + \ldots - x_{n+1}^2$ for all $(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}$; $c = -1, 0, 1$.

6. (a) Show that the graph of any function $f: \mathbf{R}^n \to \mathbf{R}$ is a level set for some function $F: \mathbf{R}^{n+1} \to \mathbf{R}$.

   (b) Give an example of a level set which is not the graph of a function. Explain why the example you give cannot be the graph of a function.

7. The aim of this question is to show that the notion of the angle between two vectors introduced in lectures, which makes sense for vectors of any dimension, coincides with the meaning of the angle between two vectors in the plane. Let $\mathbf{v} = (p, v) = (p, v_1, v_2)$ and $\mathbf{w} = (p, w) = (p, w_1, w_2)$ be two vectors based at the point $p \in \mathbf{R}^2$ and denote by θ the angle between them. Use your knowledge of trigonometry (or any other method) to prove that

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos \theta,$$

   where $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$ and $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.