F11PG2 **May 2011**

Geometry

Comment: *All definitions and theorems have been given in lectures (except Question 5). The remaining content of Questions 2 and 4 have been seen in homework. The remaining content of Question 1 and 3(b) is similar to given homework questions and 3(d) was done in lectures. Question 5 is from Chapter 11 of Thorpe, which was given as reading to the fifth years, (a) and (b) are stated and proved in that chapter, (c) is an expanded version of a homework question in that chapter.*

1.

(a) Given a smooth vector field $\mathbf{X}$ on an open set $U \subseteq \mathbf{R}^{n+1}$, define the notion of an integral curve of $\mathbf{X}$. **[3 marks]**

(b) State precisely a theorem regarding the existence and uniqueness of maximal integral curves of a smooth vector field $\mathbf{X}$ on an open set $U \subseteq \mathbf{R}^{n+1}$ through a point $p \in U$. **[7 marks]**

(c) A vector field $\mathbf{X}$ on $\mathbf{R}^2$ is defined by $\mathbf{X}(q) = (q, -q/3)$ for all $q \in \mathbf{R}^2$.

(i) Sketch the vector field $\mathbf{X}$. **[2 marks]**

(ii) Show that finding an integral curve $\alpha \colon I \to \mathbf{R}^2$ of $\mathbf{X}$ through $p \in \mathbf{R}^2$ is equivalent to solving the first order system

$$\begin{cases} x'(t) = -x(t)/3 \\ y'(t) = -y(t)/3 \end{cases}$$

subject to the initial conditions $x(0) = p_1$ and $y(0) = p_2$, where $p = (p_1, p_2)$. **[3 marks]**

(iii) Either by solving the system above, or by some other method, find the maximal integral curve of $\mathbf{X}$ through p for all $p \in \mathbf{R}^2$. **[3 marks]**

(iv) Is the vector field $\mathbf{X}$ complete? Justify your answer. **[2 marks]**

Solution:

(a) A parametrised curve $\alpha \colon I \to \mathbf{R}^{n+1}$ **(1 mark)** is said to be an *integral curve* of the vector field $\mathbf{X}$ on the open set $U \subset \mathbf{R}^{n+1}$ if $\alpha(t) \in U$ **(1 mark)** and $\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$ **(1 mark)** for all $t \in I$. Thus α has the property that its velocity vector at each point coincides with the value of the vector field $\mathbf{X}$ at the same point.

(b) **Theorem.** Let $\mathbf{X}$ be a smooth vector field on an open set $U \subset \mathbf{R}^{n+1}$ and let $p \in U$ **(1 mark)**. Then there exists an open interval I containing 0 and an integral curve $\alpha \colon I \to U$ of $\mathbf{X}$ such that **(2 marks)**

(i) $\alpha(0) = p$ **(2 marks)**, and

(ii) If $\beta \colon \widetilde{I} \to U$ is any other integral curve of $\mathbf{X}$ (for some open interval $\widetilde{I}$) with $\beta(0) = p$, then $\widetilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \widetilde{I}$ **(2 marks)**.

The integral curve α is called the *maximal integral curve* of $\mathbf{X}$ through p, or *the* integral curve of $\mathbf{X}$ through p, for short **(0 marks)**.

(c) (i) Arrows pointing towards the origin **(1 mark)** decreasing in length (linearly) as they get closer to the origin **(1 mark)**.

(ii) The parametrised curve $\alpha\colon I \to \mathbf{R}^2$ needs to satisfy $\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$ with $0 \in I$ and $\alpha(0) = p$ **(1 mark)**. Writing out the components of this with $\alpha = (x, y)$ gives

$$\begin{cases} x'(t) = -x(t)/3 \\ y'(t) = -y(t)/3 \end{cases}$$

with $x(0) = p_1$ and $y(0) = p_2$, where $p = (p_1, p_2)$ **(2 marks)**.

(iii) Using, for example, matrix exponentials or simply inspection, we can see that $x(t) = ae^{-t/3}$ and $y(t) = be^{-t/3}$, for constants $a, b \in \mathbf{R}$ **(2 marks)**. The initial condition means that $a = p_1$ and $b = p_2$, so $x(t) = p_1e^{-t/3}$ and $y(t) = p_2e^{-t/3}$ for all $t \in \mathbf{R}$ and hence $\alpha(t) = (p_1e^{-t/3}, p_2e^{-t/3})$ for all $t \in \mathbf{R}$ **(1 mark)**.

(iv) Since α found above has domain equal to $\mathbf{R}$ for each $p \in \mathbf{R}^2$ **(1 mark)**, $\mathbf{X}$ is complete **(1 mark)**.

2.

(a) Define what it means for S to be an n-surface in $\mathbf{R}^{n+1}$. **[4 marks]**

(b) For $p \in S$, state the definition of S_p, the tangent space of S at p. **[4 marks]**

(c) The set $\mathbf{R}^4$ may be viewed as the set of all 2×2 matrices with real entries by identifying the quadruple (x_1, x_2, x_3, x_4) with the matrix

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

The subset consisting of those matrices with determinant equal to one forms a group under matrix multiplication, this group is called the special linear group $SL(2)$.

(i) Show that $SL(2)$ is a 3-surface in $\mathbf{R}^4$. [Hint: The determinant of the matrix above is $x_1x_4 - x_2x_3$.] **[5 marks]**

(ii) The trace of a matrix

$$A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$$

is defined to be $\mathrm{tr}(A) := x_1 + x_4$. Show that the tangent space $SL(2)_p$ to $SL(2)$ at $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ can be identified with the set of all 2×2 matrices of trace zero by showing that

$$SL(2)_p = \left\{ \left(p, \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right) \,\middle|\, x_1 + x_4 = 0 \right\}.$$

[7 marks]

Solution:

(a) A *surface of dimension* n or an n*-surface* in $\mathbf{R}^{n+1}$ is a non-empty subset S of $\mathbf{R}^{n+1}$ **(1 mark)** of the form $S = f^{-1}(c)$ **(1 marks)**, where $f\colon U \to \mathbf{R}$ is a smooth function on an open set $U \subset \mathbf{R}^{n+1}$ with the property that $\nabla f(p) \neq \mathbf{0}$ for all $p \in S$ **(2 marks)**.

(b) The tangent space S_p is the set of all vectors $\mathbf{v} \in \mathbf{R}^{n+1}_p$ that are velocity vectors of parametrised curves in S **(2 marks)** passing through p **(2 marks)**. Alternatively, $S_p = (\nabla f(p))^\perp$ by Theorem 3.4 **(4 marks)**.

(c) (i) Set

$$f(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$$

for all $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$. Then $SL(2) = f^{-1}(1)$ **(2 marks)** and we must check that $\nabla f(x_1, x_2, x_3, x_4) \neq \mathbf{0}$ for all $(x_1, x_2, x_3, x_4) \in SL(2)$ **(1 mark)**. Indeed,

$$\nabla f(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, x_4, -x_3, -x_2, x_1)$$

which is zero exactly when $x_4 = x_3 = x_2 = x_1 = 0$. As $(0, 0, 0, 0) \notin f^{-1}(1) = SL(2)$, we have that $\nabla f(x_1, x_2, x_3, x_4) \neq \mathbf{0}$ for all $(x_1, x_2, x_3, x_4) \in SL(2)$, as required **(2 marks)**.

(ii) We know that $f(\alpha(t)) = 1$ **(1 mark)**, so

$$0 = (f \circ \alpha)'(t) = \alpha_1'(t)\alpha_4(t) + \alpha_1(t)\alpha_4'(t) - \alpha_2'(t)\alpha_3(t) - \alpha_2(t)\alpha_3'(t)$$

(2 marks) and consequently

$$0 = (f \circ \alpha)'(0) = \alpha_1'(0) + \alpha_4'(0).$$

(1 mark) From this we conclude a tangent vector of $SL(2)$ at $p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ can be identified with a 2×2 matrix of trace zero **(1 mark)**. However, we know that $SL(2)_p$ is a three dimensional vector space, since $SL(2)$ is a 3-surface, and 2×2 matrices of trace zero are also a three dimensional vector space (under addition of matrices and multiplication by scalars), so $SL(2)_p$ equals the set of 2×2 matrices of trace zero **(2 marks)**.

3.

(a) Define the Gauss map for an n-surface S with orientation $\mathbf{n}$. **[2 marks]**

(b) Sketch the image of the Gauss map N for the 1-surface $f^{-1}(0)$ with orientation $\nabla f / \|\nabla f\|$ when f is given as follows.

(i) $f(x_1, x_2) = x_2 - x_1^2$ for all $(x_1, x_2) \in \mathbf{R}^2$, and **[2 marks]**

(ii) $f(x_1, x_2) = x_1$ for all $(x_1, x_2) \in \mathbf{R}^2$. **[2 marks]**

(c) State precisely a theorem regarding the Gauss map that gives a condition under which it is surjective. **[5 marks]**

(d) Let $S = f^{-1}(0)$ be an oriented n-surface with orientation

$$\mathbf{n} = (\cdot, N(\cdot)) = \nabla f / \|\nabla f\|$$

for some smooth $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$, let $p_1, p_2 \in S$ and $q \in \mathbf{S}^n$. Suppose that there exists a continuous function $\alpha\colon [a,b] \to \mathbf{R}^{n+1}$, differentiable at a and b, such that

(i) $\alpha(a) = p_1$, $\alpha(b) = p_2$, $\dot{\alpha}(a) = (p_1, q)$ and $\dot{\alpha}(b) = (p_2, q)$, and

(ii) $\alpha(t) \notin S$ for $a < t < b$.

Prove that $N(p_1) \neq N(p_2)$. [Hint: Consider $f \circ \alpha$.] **[9 marks]**

Solution:

(a) The Gauss map is the 'arrow part' of $\mathbf{n}$. That is for $\mathbf{n} = (\cdot, N(\cdot))$, the Gauss map is $N\colon S \to \mathbf{S}^n$. **(2 marks)**

(b) (i) It is the 'northern hemisphere' of $\mathbf{S}^1$ **(2 marks)**.

(ii) It is the point 'due east' from the centre of $\mathbf{S}^1$ **(2 marks)**.

(c) **Theorem.** Let S be a compact **(3 marks)** connected **(2 marks)** oriented n-surface in $\mathbf{R}^{n+1}$ (so we can write $S = f^{-1}(c)$ for some smooth function $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$ with $\nabla f(p) \neq \mathbf{0}$ for all $p \in S$ and $c \in \mathbf{R}$) with orientation $\mathbf{n} = (\cdot, N(\cdot))$. Then the Gauss map N maps S onto the n-sphere (that is, the Gauss map $N\colon S \to \mathbf{S}^n$ is surjective).

(d) By (i), we have that

$$(f \circ \alpha)'(a) = \nabla f(\alpha(a)) \cdot \dot{\alpha}(a) = \|\nabla f(p_1)\| \mathbf{n}(p_1) \cdot (p_1, q),$$

(2 marks) and, similarly

$$(f \circ \alpha)'(b) = \|\nabla f(p_2)\| \mathbf{n}(p_2) \cdot (p_2, q)$$

(1 mark). Thus, if $N(p_1) = N(p_2)$, then $\mathbf{n}(p_1) \cdot (p_1, q) = \mathbf{n}(p_2) \cdot (p_2, q)$, and so (by the above) $(f \circ \alpha)'$ has the same sign at both end points. From this it follows that $(f \circ \alpha)$ is either increasing at both end-points or decreasing at both end points **(2 mark)**. Consequently, using the fact that $(f \circ \alpha)(a) = (f \circ \alpha)(b) = 0$, we know there exist $t_1, t_2 \in (a,b)$ such that $(f \circ \alpha)(t_1) > 0$ and $(f \circ \alpha)(t_2) < 0$ **(2 marks)**, and so by the Intermediate Value Theorem, there exists a number $t_3 \in (a,b)$ such that $(f \circ \alpha)(t_3) = 0$ **(2 marks)**. This contradicts (ii), so it must be that $N(p_1) \neq N(p_2)$.

4.

(a) Let S be an oriented n-surface with orientation $\mathbf{n}$. Define the Weingarten map L_p for a $p \in S$. **[5 marks]**

(b) State precisely a theorem which relates the value of the Weingarten map $L_p(\mathbf{v})$ at $\mathbf{v} \in S_p$ and the acceleration of a parametrised curve $\alpha\colon I \to S$ with velocity $\dot{\alpha}(t_0) = \mathbf{v}$. **[7 marks]**

(c) Show that all integral curves of a smooth tangent vector field $\mathbf{X}$ on S are geodesics if and only if $\nabla_{\mathbf{X}(p)}\mathbf{X}(p) \perp S_p$ for all $p \in S$. **[8 marks]**

Solution:

(a) The linear map $L_p \colon S_p \to S_p$ defined by

$$L_p(\mathbf{v}) = -\nabla_{\mathbf{v}}\mathbf{n}(p)$$

is called the *Weingarten map* of the oriented n-surface S (with orientation $\mathbf{n}$) at p **(5 marks)**.

(b) **Theorem.** Let S be an n-surface in $\mathbf{R}^{n+1}$ with orientation $\mathbf{n}$. Let $p \in S$ and $\mathbf{v} \in S_p$. Then for every parametrised curve $\alpha \colon I \to S$ with $\dot{\alpha}(t_0) = \mathbf{v}$ for some $t_0 \in I$ (and in particular $\alpha(t_0) = p$) **(3 marks)**,

$$\ddot{\alpha}(t_0) \cdot \mathbf{n}(p) = L_p(\mathbf{v}) \cdot \mathbf{v}$$

(4 marks).

(c) An integral curve of $\mathbf{X}$ through p is a parametrised curve such that $\alpha(0) = p$ and $\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$ for all t in the domain of α, so

$$\ddot{\alpha}(t) = (\mathbf{X} \circ \alpha)^{\cdot}(t) = \nabla_{\mathbf{X}(\alpha(t))}\mathbf{X}(\alpha(t))$$

(2 marks). Therefore $\ddot{\alpha}(t) \perp S_{\alpha(t)}$ if and only if $\nabla_{\mathbf{X}(\alpha(t))}\mathbf{X}(\alpha(t)) \perp S_{\alpha(t)}$ **(2 marks)**. Consequenly, if $\nabla_{\mathbf{X}(p)}\mathbf{X}(p) \perp S_p$ for all $p \in S$, then every integral curve α is a geodesic **(2 marks)**. Conversely, for fixed $p \in S$ let α be the integral curve through p, then $\ddot{\alpha}(0) = \nabla_{\mathbf{X}(p)}\mathbf{X}(p)$, and if α is a geodesic $\nabla_{\mathbf{X}(p)}\mathbf{X}(p) \perp S_p$ **(2 marks)**.

5.

The length $\ell(\alpha)$ of a parametrised curve $\alpha \colon (a, b) \to \mathbf{R}^{n+1}$ is defined to be

$$\ell(\alpha) = \int_a^b \|\dot{\alpha}(t)\| dt,$$

where $-\infty < a < b < \infty$.

(a) For $-\infty < c < d < \infty$, suppose that $\beta = \alpha \circ h$, where $h \colon [c, d] \to [a, b]$ is a continuous function such that $h'(t) > 0$ for all $t \in (c, d)$, $h(c) = a$ and $h(d) = b$. Show that $\ell(\alpha) = \ell(\beta)$. **[6 marks]**

(b) State precisely a theorem which gives a dichotomy of unit speed global parametrisations of connected oriented plane curves. Which alternative occurs for compact plane curves? **[5 marks]**

(c) Let C be a connected oriented plane curve with orientation $\mathbf{n}$, let $\alpha \colon I \to C$ be a one-to-one unit speed parametrisation of C and let $\kappa \colon C \to \mathbf{R}$ denote curvature on C. Show that

$$(\kappa \circ \alpha)(t) = -\dot{\alpha} \cdot (\mathbf{n} \circ \alpha)^{\cdot}(t)$$

and use this to show that

$$\int_a^b |(\kappa \circ \alpha)(t)| dt = \ell(N \circ \alpha),$$

where a and b are the end-points of I, and $N\colon C \to \mathbf{S}^1 \subset \mathbf{R}^2$ is the Gauss map of C. **[9 marks]**

Solution:

(a)

$$\begin{aligned}
\ell(\beta) &= \int_c^d \|\dot\beta(t)\| dt = \int_c^d \|(\alpha \circ h)^{\cdot}(t)\| dt \\
&= \int_c^d \|\dot\alpha(h(t))h'(t)\| dt = \int_c^d \|\dot\alpha(h(t))\| h'(t) dt \\
&= \int_a^b \|\dot\alpha(t)\| dt = \ell(\alpha).
\end{aligned}$$

(6 marks)

(b) **Theorem.** Let C be a connected oriented plane curve and let $\beta\colon I \to C$ be a unit speed global parametrisation of C. Then β is either one-to-one or periodic **(3 marks)**. Moreover, β is periodic if and only if C is compact **(2 marks)**.

(c) First

$$(\kappa \circ \alpha)(t) = \ddot\alpha(t) \cdot \mathbf{n}(\alpha(t))$$

(2 marks) and

$$\ddot\alpha(t) \cdot \mathbf{n}(\alpha(t)) = (\dot\alpha \cdot (\mathbf{n} \circ \alpha))'(t) - \dot\alpha \cdot (\mathbf{n} \circ \alpha)^{\cdot}(t) = -\dot\alpha \cdot (\mathbf{n} \circ \alpha)^{\cdot}(t),$$

(1 mark) where $\mathbf{n}$ is the orientation of C, so

$$(\kappa \circ \alpha)(t) = -\dot\alpha \cdot (\mathbf{n} \circ \alpha)^{\cdot}(t).$$

But

$$-(\mathbf{n} \circ \alpha)^{\cdot}(t) = -\nabla_{\dot\alpha(t)} \mathbf{n}(\alpha(t)) = L_{\alpha(t)}(\dot\alpha(t)) \in C_{\alpha(t)}.$$

(2 marks) Moreover, $C_{\alpha(t)}$ is 1-dimensional, $\dot\alpha(t) \in C_{\alpha(t)}$ and $\|\dot\alpha(t)\| = 1$, so $|-\dot\alpha \cdot (\mathbf{n} \circ \alpha)^{\cdot}(t)| = \|\nabla_{\dot\alpha(t)} \mathbf{n}(\alpha(t))\|$ **(2 marks)**. Finally,

$$\nabla_{\dot\alpha(t)} \mathbf{n}(\alpha(t)) = (N \circ \alpha)^{\cdot}(t),$$

(1 mark) so

$$\begin{aligned}
\ell(N \circ \alpha) &= \int_a^b \|(N \circ \alpha)^{\cdot}(t)\| dt \\
&= \int_a^b \|\nabla_{\dot\alpha(t)} \mathbf{n}(\alpha(t))\| dt = \int_a^b |-\dot\alpha \cdot (\mathbf{n} \circ \alpha)^{\cdot}(t)| dt \\
&= \int_a^b |(\kappa \circ \alpha)(t)| dt.
\end{aligned}$$

(1 mark).
