Fourier Analysis
Second Semester 2008/9
Mock Examination

The following questions are of the style you can expect in the final examination.

1. Let f be the characteristic function of the interval $[a, b] \subset [-\pi, \pi]$.
   (a) State Jordan's criterion for the convergence of Fourier series.
   (b) Show that the Fourier series of f is
   $$\frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina} - e^{inb}}{2\pi i n} e^{inx}.$$
   (c) Prove that the above Fourier series converges for each value of x and determine the function to which it converges.
2. Suppose f is 2π-periodic and integrable on $[-\pi, \pi]$.
   (a) Show that
   $$\hat{f}(n) = \frac{1}{4\pi} \int_{-\pi}^{\pi} [f(x) - f(x + \pi/n)] e^{-inx}\, dx.$$
   (b) Assume that f is Hölder continuous of order α, that is
   $$|f(x) - f(y)| \leq C|x-y|^{\alpha}$$
   for some $C > 0$ and all x and y. Use part (a) to prove
   $$\hat{f}(n) = O(1/|n|^{\alpha}).$$
3. (a) Define the convolution $f * g$ of two functions f and g defined on $\mathbf{R}$.
   (b) Prove that $(f * g)\widehat{\ }(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$.
   (c) Define
   $$f(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$
   and show by direct calculation that $\widehat{f}(\xi) = \sin(2\pi\xi)/(\pi\xi)$. Use the above to find a function h for which
   $$\widehat{h}(\xi) = \left(\frac{\sin(2\pi\xi)}{\pi\xi}\right)^2.$$
4. Let f be a function of moderate decrease. This question illustrates the principle that decay in the Fourier transform $\widehat{f}$ is related to the continuity of the function f. Suppose that $\widehat{f}$ is continuous and satisfies
   $$|\widehat{f}(\xi)| \leq \frac{C}{|\xi|^{1+\alpha}}$$
   for some $C > 0$ and $0 < \alpha < 1$.
   (a) Derive an expression for $f(x+h) - f(x)$ using the inversion formula for the Fourier transform. This expression should be an integral of some kind.
   (b) Split the integral in two, integrating over $|\xi| \leq 1/|h|$ and $|\xi| > 1/|h|$. Estimate each integral separately to conclude
   $$|f(x+h) - f(x)| \leq A|h|^{\alpha}$$
   for some $A > 0$.