Fourier Analysis
Second Semester 2008/9
Homework Assignment 4
(Due on 6th March 2009)

Only the questions marked with an asterisk (*) will count towards the assessment for this course. Most of these exercises are taken from Stein and Shakarchi.

1. Prove the following proposition, which we stated in class.

   Proposition. *If $f \in \mathcal{S}(\mathbf{R})$ then*
   (i) $f(x+h) \to \widehat{f}(\xi)e^{2\pi ih\xi}$ *whenever* $h \in \mathbf{R}$,
   (ii) $f(x)e^{-2\pi ixh} \to \widehat{f}(\xi+h)$ *whenever* $h \in \mathbf{R}$,
   (iii) $f(\delta x) \to \delta^{-1}\widehat{f}(\delta^{-1}\xi)$ *for* $\delta > 0$,
   (iv) $f'(x) \to 2\pi i\xi\widehat{f}(\xi)$,
   (v) $-2\pi ixf(x) \to (\widehat{f})'(\xi)$.

*2. The aim of this question is to prove that $g\colon \mathbf{R} \to \mathbf{C}$ defined by

$$g(x) = e^{-\pi x^2}$$

for all $x \in \mathbf{R}$ is equal to its own Fourier transform.
   (a) By a simple calculation, check that $g'(x) = -2\pi xg(x)$.
   (b) Set

$$G(\xi) = \widehat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{-2\pi ix\xi}\,dx$$

   and use the above fact to prove that $G'(\xi) = -2\pi\xi G(\xi)$.
   (c) Define $H(\xi) = G(\xi)e^{\pi\xi^2}$ and show that $H'(\xi) = 0$ for all $\xi \in \mathbf{R}$.
   (d) Conclude that $G(\xi) = e^{-\pi\xi^2}$. You may assume $G(0) = 1$ without proof.

*3. We have proved the following theorem in class.

   Theorem. *If a continuous function f on $[-\pi,\pi]$ (such that $f(-\pi) = f(\pi)$) has a Fourier series which is absolutely convergent, then the Fourier series converges to the function. That is,*

$$\lim_{N\to\infty} \sum_{n=-N}^{N} a_n e^{inx} = f(x)$$

   for all $x \in [-\pi,\pi]$, where

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

   Suppose now that $f\colon \mathbf{R} \to \mathbf{C}$ is zero outside an interval $[-M,M]$ and its Fourier transform $\widehat{f}$ is of moderate decrease. The aim of this question is to use the above theorem to find a simpler proof of the Fourier inversion formula for such functions f. Throughout this question $\widehat{f}$ denotes the Fourier transform of f not the Fourier coefficients (the formal difference between the two is only the value of certain constants).
   (a) Fix L such that $L > 2M$ and show that $f(x) = \sum_n a_n(L)e^{2\pi inx/L}$, where

$$a_n(L) = \frac{1}{L}\int_{-L/2}^{L/2} f(x)e^{-2\pi inx/L}\,dx = \frac{\widehat{f}(n/L)}{L}.$$

   This can be thought of as the theorem above rewritten for a function on $[-L/2, L/2]$ (Observe that $a_n = a_n(2\pi)$). Show this can also be stated as

$$f(x) = \delta\sum_{n=-\infty}^{\infty}\widehat{f}(n\delta)e^{2\pi inx/L}$$

   with $\delta = 1/L$.
   (b) Prove that if F is continuous and of moderate decrease, then

$$\int_{-\infty}^{\infty} F(\xi)\,d\xi = \lim_{\delta\searrow 0}\delta\sum_{n=-\infty}^{\infty}F(\delta n).$$

   [Hint: Approximate the integral by Riemann sums.]
   (c) Conclude that $f(x) = \int_{-\infty}^{\infty}\widehat{f}(\xi)e^{2\pi in\xi}\,d\xi$.

4. Let f and g be functions defined by

$$f(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$

and

$$g(x) = \begin{cases} 1 - |x|, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$

Although f is not continuous, observe the integral defining the Fourier transform of f still makes sense. Show that

$$\widehat{f}(\xi) = \frac{\sin(2\pi\xi)}{\pi\xi} \quad \text{and} \quad \widehat{g}(x) = \left(\frac{\sin(2\pi\xi)}{\pi\xi}\right)^2,$$

for $\xi \neq 0$ and compute $\widehat{f}(0)$ and $\widehat{g}(0)$.

*5. Let f be a function of moderate decrease. This question illustrates the principle that decay in the Fourier transform $\widehat{f}$ is related to the continuity of the function f. Suppose that $\widehat{f}$ is continuous and satisfies

$$|\widehat{f}(\xi)| \leq \frac{C}{|\xi|^{1+\alpha}}$$

for some $C > 0$ and $0 < \alpha < 1$.

(a) Derive an expression for $f(x+h) - f(x)$ using the inversion formula for the Fourier transform. This expression should be an integral of some kind.

(b) Split the integral in two, integrating over $|\xi| \leq 1/|h|$ and $|\xi| > 1/|h|$. Estimate each integral separately to conclude

$$|f(x+h) - f(x)| \leq A|h|^\alpha$$

for some $A > 0$.

6. Suppose f is a function of moderate decrease (hence, also continuous).

(a) Prove that $\widehat{f}$ is continuous and that $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$ by proving the formula

$$\widehat{f}(\xi) = \frac{1}{2}\int [f(x) - f(x - 1/(2\xi)]e^{-2\pi i x\xi}\, d\xi.$$

(b) Show that if $\widehat{f}(\xi) = 0$ for all $\xi \in \mathbf{R}$, then $f(x) = 0$ for all $x \in \mathbf{R}$. [Hint: Check the multiplication formula $\int \widehat{f}(y)g(y)\,dy = \int f(y)\widehat{g}(y)\,dy$ is valid for such f when $g \in \mathcal{S}(\mathbf{R})$.]