**Fourier Analysis**
**Second Semester 2008/9**
Homework Assignment 3
(Due on 24th February 2009)

Only the questions marked with an asterisk (*) will count towards the assessment for this course. Most of these exercises are taken from Stein and Shakarchi.

1. Prove that $l^2(\mathbf{Z})$ is complete, that is, every Cauchy sequence in $l^2(\mathbf{Z})$ has a limit in $l^2(\mathbf{Z})$.
2. This question follows on from question 3 on Assignment 2. Consider again the function $f\colon [-\pi, \pi] \to \mathbf{C}$ defined by $f(\theta) = |\theta|$. Use Parseval's Identity to prove
$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$
In fact, it is possible to calculate a formula for $\sum_n 1/n^k$ for every even $k$. However, nobody has yet managed to do this for odd $k$, even for $k = 3$. If you have some spare time, perhaps you might want to try to evaluate $\sum_n 1/n^3$.

*3. Suppose that $f$ is a continuously differentiable $2\pi$-periodic function such that
$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0.$$
(a) Use Parseval's Identity to show that
$$\int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta \le \int_{-\pi}^{\pi} |f'(\theta)|^2\, d\theta.$$
(b) Find a function $f_0$ which satisfies the same conditions as $f$ but
$$\int_{-\pi}^{\pi} |f_0(\theta)|^2\, d\theta = \int_{-\pi}^{\pi} |f_0'(\theta)|^2\, d\theta.$$
(c) Find a function $f_1$ which satisfies the same conditions as $f$ but
$$\int_{-\pi}^{\pi} |f_1(\theta)|^2\, d\theta < \int_{-\pi}^{\pi} |f_1'(\theta)|^2\, d\theta.$$
(d) Suppose $g$ is continuously differentiable and $f$ is as above. Prove that
$$\left| \int_{-\pi}^{\pi} \overline{f(\theta)} g(\theta)\, d\theta \right|^2 \le \int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta \int_{-\pi}^{\pi} |g'(\theta)|^2\, d\theta.$$
(e) Suppose $h$ is continuously differentiable on $[0, \pi]$ and $h(0) = h(\pi) = 0$. Prove
$$\int_{0}^{\pi} |h(\theta)|^2\, d\theta \le \int_{0}^{\pi} |h'(\theta)|^2\, d\theta.$$

*4. The aim of this question is to evaluate the improper Riemann integral
$$\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\, dx.$$
(a) Show that
$$\lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\, dx = \lim_{N\to\infty} \int_0^\pi \frac{\sin((N+1/2)x)}{x}\, dx.$$
(b) Use the Riemann-Lebesgue Lemma to prove
$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left( \frac{1}{\sin(x/2)} - \frac{2}{x} \right) \sin((N+1/2)x)\, dx = 0.$$
(c) Use the above and the fact that
$$\int_{-\pi}^{\pi} \frac{\sin((N+1/2)x)}{\sin(x/2)}\, dx = 2\pi$$
for all $N \ge 1$ to show that $\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2$.

5. Suppose $f$ is $2\pi$-periodic function which is $k$-times continuously differentiable. Prove that
$$\hat{f}(n) = O(1/|n|^k)$$
as $|n| \to \infty$.

*6. Suppose $f$ is $2\pi$-periodic and integrable on $[-\pi, \pi]$.

(a) Show that

$$\hat{f}(n) = \frac{1}{4\pi} \int_{-\pi}^{\pi} [f(x) - f(x + \pi/n)] e^{-inx}\, dx.$$

(b) Assume that $f$ is Hölder continuous of order $\alpha$, that is

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}$$

for some $C > 0$ and all $x$ and $y$. Use part (a) to prove

$$\hat{f}(n) = O(1/|n|^{\alpha}).$$