Fourier Analysis
Second Semester 2008/9
Homework Assignment 2
(Due on 6th February 2009)

Only the questions marked with an asterisk (*) will count towards the assessment for this course. Most of these exercises are taken from Stein and Shakarchi.

1. Recall the Dirichlet kernel:

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(x/2)}.$$

Show that D_N is an even function and that

$$\int_{-\pi}^{\pi} D_N(x)\,dx = 1.$$

2. For $\delta \in (0,\pi)$, let f be defined on $[-\pi,\pi]$ by

$$f(\theta) = \begin{cases} 0, & \text{if } |\theta| > \delta, \\ 1 - |\theta|/\delta, & \text{if } |\theta| \leq \delta. \end{cases}$$

(a) Plot the graph of f.
(b) Show that

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1-\cos(n\delta)}{n^2\pi\delta}\cos(n\theta).$$

*3. Let f be the function given by $f(\theta) = |\theta|$ for $\theta \in [-\pi,\pi]$.
(a) Draw the graph of f.
(b) Calculate the Fourier coefficients of f and show that

$$\hat{f}(n) = \begin{cases} \pi/2, & \text{if } n = 0, \\ \{(-1)^n - 1\}/(\pi n^2), & \text{if } n \neq 0. \end{cases}$$

(c) What is the Fourier series of f in terms of sines and cosines?
(d) Taking $\theta = 0$, prove that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

*4. Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sum of the series $\sum_n b_n$ for $k \geq 1$ and define $B_0 = 0$.
(a) **Summation by Parts.** Prove the formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

(b) Deduce from the above formula the following lemma:
Lemma (Dirichlet's convergence test). If the partial sums of the series $\sum_n b_n$ are bounded (that is, $B_k \leq C$ for all $k \geq 1$) and $\{a_n\}_n$ is a sequence of real numbers that decrease monotonically to 0, then $\sum_n a_n b_n$ converges.

*5. Consider the function $f\colon [-\pi,\pi] \to \mathbf{C}$ defined by

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & \text{if } -\pi \leq x < 0, \\ 0, & \text{if } x = 0, \\ \frac{\pi}{2} - \frac{x}{2}, & \text{if } 0 < x \leq \pi. \end{cases}$$

Draw the graph of f. Prove that the Fourier series of f is

$$\frac{1}{2i}\sum_{n\neq 0} \frac{e^{inx}}{n}$$

and prove that it converges pointwise to f, even though f is not continuous. [Hint: Use Dirichlet's convergence test.]

6. Suppose that $\{f_n\}_n$ is a sequence of integrable functions on $[-\pi,\pi]$ such that

$$\int_{-\pi}^{\pi} |f_k(x) - f(x)|\,dx \to 0$$

as $k \to \infty$ for another integrable function $f\colon [-\pi,\pi] \to \mathbf{C}$. Show that $\hat{f}_k(n) \to \hat{f}(n)$ uniformly in n as $k \to \infty$.