Fourier Analysis
Second Semester 2008/9
Solutions to Selected Homework Questions

These solutions are only sketches. Any gaps must be completed to produce the full solution.

#2, 2. For $\delta \in (0, \pi)$, we have defined f on $[-\pi, \pi]$ to be

$$f(\theta) = \begin{cases} 0, & \text{if } |\theta| > \delta, \\ 1 - |\theta|/\delta, & \text{if } |\theta| \leq \delta. \end{cases}$$

(a) Graphs of functions should be of the correct shape and have the correct position, and the axes should be labelled.

(b) In order to prove

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1 - \cos(n\delta)}{n^2\pi\delta} \cos(n\theta).$$

we will first compute the Fourier series of f and observe it equals the right-hand side of the above, and then we will prove that f is equal to its Fourier series.

Now,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{\delta}^{\delta} (1 - |\theta|/\delta) e^{in\theta}\, d\theta,$$

so $\hat{f}(0) = \delta/2\pi$ and for $n \neq 0$,

$$\hat{f}(n) = \frac{1}{\pi\delta} \int_0^{\delta} (1 - \theta) e^{in\theta}\, d\theta = \frac{1}{in\pi\delta} \int_0^{\delta} e^{in\theta}\, d\theta + \left[\frac{1}{in\pi} (1 - \theta/\delta) e^{in\theta} \right]_0^{\delta}$$

$$= \left[\frac{-1}{n^2\pi\delta} e^{in\theta} + \frac{1}{in\pi} (1 - \theta/\delta) e^{in\theta} \right]_0^{\delta} = \frac{1 - e^{in\delta}}{n^2\pi\delta} + \frac{i}{n\pi}.$$

Therefore the Fourier series can formally be written as

$$\frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1 - \cos(n\delta)}{n^2\pi\delta} \cos(n\theta),$$

since f is an even function.

Observe that the Fourier series are not absolutely convergent, so we are not able to conclude the Fourier series converges uniformly to f. However, f is continuous and locally of bounded variation (for example, $f = f_1 + f_2$ where

$$f_1(\theta) = \begin{cases} 0, & \text{if } \theta < -\delta, \\ 1 + \theta/\delta, & \text{if } -\delta\theta \leq 0, \\ 1, & \text{if } 0 < \theta \end{cases}$$

is non-decreasing and f_2 is non-increasing), so we can apply Jordan's criterion to conclude that the Fourier series converges to f at every θ.

#2, 6. We compute

$$|\hat{f}_k(n) - \hat{f}(n)| = \left| \frac{1}{2\pi} \int_{\pi}^{\pi} (f_k(x) - f(x)) e^{inx}\, dx \right| \leq \frac{1}{2\pi} \int_{\pi}^{\pi} |f_k(x) - f(x)|\, dx \to 0$$

as $k \to \infty$.

#3, 1. Suppose $A_n = \{a_n^N\}_n$ is a Cauchy sequence in $l^2(\mathbf{Z})$. Therefore, for each $n \in \mathbf{Z}$,

$$|a_n^N - a_n^M| \leq \left(\sum_{m \in \mathbf{Z}} |a_m^N - a_m^M|^2 \right)^{\frac{1}{2}} \to 0$$

as $M, N \to \infty$, so $\{a_n^N\}_N$ is a Cauchy sequence in $\mathbf{R}$. Therefore, by the completeness of $\mathbf{R}$, we can define a component-wise limit of A_N to be $A = \{a_n\}_n$, where $a_n = \lim_{N\to\infty} a_n^N$.

Now fix $\varepsilon > 0$. There exists an number N_0 such that, for $N, M > N_0$, we have

$$\left(\sum_{m \in \mathbf{Z}} |a_m^N - a_m^M|^2 \right)^{\frac{1}{2}} < \varepsilon.$$

Now, by Fatou's lemma,

$$\left(\sum_{n \in \mathbf{Z}} |a_n^N - a_n|^2 \right)^{\frac{1}{2}} \leq \liminf_{M\to\infty} \left(\sum_{m \in \mathbf{Z}} |a_m^N - a_m^M|^2 \right)^{\frac{1}{2}} < \varepsilon,$$

for $N > N_0$. This proves both that $A \in l^2(\mathbf{Z})$ and that $\lim_{N\to\infty} A_N = A$ in $l^2(\mathbf{Z})$.

#4, 4. Clearly $\widehat{f}(0) = 1$. Also, for $\xi \neq 0$,

$$\widehat{f}(\xi) = \int_{-1}^{1} e^{2\pi i x\xi}\, dx = \left[\frac{e^{2\pi i x\xi}}{2\pi i\xi}\right]_{-1}^{1} = \frac{e^{2\pi i\xi}}{2\pi i\xi} - \frac{e^{-2\pi i\xi}}{2\pi i\xi} = \frac{\sin(2\pi\xi)}{\pi\xi}.$$

Computing the Fourier transform of g is similar.

#4, 6. We have

$$\widehat{f}(\xi) - \widehat{f}(\xi + h) = (1 - e^{2\pi i h\xi})\widehat{f}(\xi) \to 0$$

as $h \to 0$, so $\widehat{f}$ is continuous. Also, using the given formula (which is easy to prove using the periodicity of $e^{2\pi i x\xi}$)

$$|\widehat{f}(\xi)| \leq \frac{1}{2}\int |f(x) - f(x - 1/(2\xi)|\, d\xi \to 0$$

as $|\xi| \to \infty$.

#5, 5. (a) We have

$$(Af, g) = \int_{\mathbf{R}} f'(x)\overline{g(x)} + xf(x)\overline{g(x)}\, dx$$
$$= \int_{\mathbf{R}} -f(x)\overline{g'(x)} + f(x)\overline{xg(x)}\, dx = (f, A^*g).$$

(b) Using (a),

$$(A^*A(f), f) = (Af, Af) = \|Af\|^2 \geq 0.$$

(c) This one was harder than I thought, sorry. Define

$$A_t f(x) = f'(x) + txf(x) \ \text{ and } \ A_t^* f(x) = -f'(x) + txf(x).$$

and observe that we can prove $(A_t^* A_t f, f) \geq 0$ just as before. Now,

$$0 \leq (A_t^* A_t f, f) = \int_{\mathbf{R}} -f''(x)\overline{f(x)} + x^2|f(x)|^2 - |f(x)|^2\, dx$$
$$= \int_{\mathbf{R}} |f'(x)|^2 + t^2x^2|f(x)|^2 - t|f(x)|^2\, dx$$

and so viewing the right-hand side as a polynomial in t, it can't have more than one real root, so we must have

$$\left(\int_{\mathbf{R}} |f(x)|^2\, dx\right)^2 - 4\left(\int_{\mathbf{R}} |f'(x)|^2\, dx\right)\left(\int_{\mathbf{R}} |f(x)|^2\, dx\right) \leq 0.$$

Rearranging this, using the normalisation condition and Plancherel's Theorem gives the uncertainty principle. The other direction is proved by reversing these steps.