Fourier Analysis
Second Semester 2009/10
Homework Assignment 4 Solutions

Question 1.

In lectures we have proved the Heisenberg uncertainty principle, which says for $\psi \in \mathcal{S}(\mathbf{R})$ such that $\|\psi\|_{L^2(\mathbf{R})} = 1$ we have that

$$\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\widehat{\psi}(\xi)|^2\,d\xi\right) \geq \frac{1}{16\pi^2}. \tag{1}$$

Use this to prove that

$$\left(\int_{-\infty}^{\infty} (x-x_0)^2|\psi(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} (\xi-\xi_0)^2|\widehat{\psi}(\xi)|^2\,d\xi\right) \geq \frac{1}{16\pi^2}$$

for each $x_0, \xi_0 \in \mathbf{R}$.

Solution. We know from Proposition 3.1.6 in the notes that

$$(\psi(\cdot + x_0))\widehat{\ }(\xi) = \widehat{\psi}(\xi)e^{2\pi i x_0 \xi} \tag{2}$$

and

$$(\phi(\cdot)e^{-2\pi i \xi_0 \cdot})\widehat{\ }(\xi) = \widehat{\phi}(\xi + \xi_0), \tag{3}$$

for any $\phi, \psi \in \mathcal{S}(\mathbf{R})$ and $x_0, \xi_0 \in \mathbf{R}$. For $\psi \in \mathcal{S}(\mathbf{R})$ we define ϕ as $\phi(x) = \psi(x + x_0)$ for all $x \in \mathbf{R}$ and a fixed $x_0 \in \mathbf{R}$. Clearly $\phi \in \mathcal{S}(\mathbf{R})$ and, using (3),

$$(\psi(\cdot + x_0)e^{-2\pi i \xi_0 \cdot})\widehat{\ }(\xi) = (\phi(\cdot)e^{-2\pi i \xi_0 \cdot})\widehat{\ }(\xi) = \widehat{\phi}(\xi + \xi_0).$$

However, by (2), $\widehat{\phi}(\xi) = (\psi(\cdot + x_0))\widehat{\ }(\xi) = \widehat{\psi}(\xi)e^{2\pi i x_0 \xi}$, so

$$(\psi(\cdot + x_0)e^{-2\pi i \xi_0 \cdot})\widehat{\ }(\xi) = \widehat{\phi}(\xi + \xi_0) = \widehat{\psi}(\xi + \xi_0)e^{2\pi i x_0(\xi + \xi_0)}. \tag{4}$$

Now we write down (1) with ψ replaced by the function $x \mapsto \psi(x + x_0)e^{-2\pi i \xi_0 x}$. Observe this function is a Schwarz function, so this step is justified. We find

$$\begin{aligned}
\frac{1}{16\pi^2} &\leq \left(\int_{-\infty}^{\infty} x^2|\psi(x+x_0)e^{-2\pi i \xi_0 x}|^2\,dx\right)\left(\int_{-\infty}^{\infty} \xi^2|(\psi(\cdot + x_0)e^{-2\pi i \xi_0 \cdot})\widehat{\ }(\xi)|^2\,d\xi\right)\\
&= \left(\int_{-\infty}^{\infty} x^2|\psi(x+x_0)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\widehat{\psi}(\xi+\xi_0)e^{2\pi i x_0(\xi+\xi_0)}|^2\,d\xi\right)\\
&= \left(\int_{-\infty}^{\infty} x^2|\psi(x+x_0)|^2\,dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\widehat{\psi}(\xi+\xi_0)|^2\,d\xi\right)\\
&= \left(\int_{-\infty}^{\infty} (x-x_0)^2|\psi(x)|^2\,dx\right)\left(\int_{-\infty}^{\infty} (\xi-\xi_0)^2|\widehat{\psi}(\xi)|^2\,d\xi\right),
\end{aligned}$$

as required.

Mark Scheme. People employed two methods. The marks for the method above were assigned as follows. Four marks for justifying (4), two marks for correctly substituting this in (1), and four marks for calculations. The second method was similar, but first students proved a more general version of (1). Strictly speaking this didn't use the Heisenberg uncertainty principle we proved in lectures, but I didn't take away marks for that. For the second method the mark scheme was four points for re-proving the uncertainty principle, four points for justifying formulae, and two points for calculation.

Notes. Some people seem confused by the dot notation we have used in class. A function f from a set D to another set R is a rule which assigns to each element $x \in D$ an element $f(x) \in R$. We write $f\colon D \to R$. Strictly speaking f is notation for the function and $f(x)$ denotes the value of the function when evaluated at some $x \in D$. Another way to describe a function is to write

$$x \mapsto f(x),$$

which means $x \in D$ is assigned the value $f(x)$. A third notation is to write a dot where the variable would be, so we write $f(\cdot)$.

Each notation has its advantages in different situations. For example, the first notation requires that we assign a label to the function (in this case f), but once this is done it is quite concise. If we define $f\colon \mathbf{R} \to \mathbf{R}$ as the function which to each $x \in \mathbf{R}$ assigns the value $f(x) = e^x \sin(x) + \cos(x)$, it is quicker to write 'f is differentiable' than '$x \mapsto e^x \sin(x) + \cos(x)$ is differentiable'. However, if we only refer to the function once, the second option might be more convenient.

You may see (or write) statements such as 'the function $e^x \sin(x) + \cos(x)$ is continuous' but this is an abuse of notation and should be avoided as it can be ambiguous. For example, if we say '$xy/(x^2+y^2)$ is continuous' it is unclear if we mean $x \mapsto xy/(x^2+y^2)$, $y \mapsto xy/(x^2+y^2)$ or $(x,y) \mapsto xy/(x^2+y^2)$ is continuous. Indeed, in this example only two of the three possibilities are true, so we may need to be precise.

We have used the dot notation when taking the Fourier transform of a function. For example, in the homework we wished to take the Fourier transform of the translation of a function, that is, we wanted to take the Fourier transform of $\phi\colon \mathbf{R} \to \mathbf{C}$ where $\phi(x) = \psi(x+y)$ for each x, a fixed $y \in \mathbf{R}$ and $\psi \in \mathcal{S}(\mathbf{R})$. Writing $(\psi(x+y))\widehat{\ }(\xi)$ is ambiguous: we may mean either 'the Fourier transform of $x \mapsto \psi(x+y)$ evaluated at ξ', or 'the Fourier transform of $y \mapsto \psi(x+y)$ evaluated at ξ', or 'the Fourier transform of the constant function $t \mapsto \psi(x+y)$ evaluated at ξ'. To avoid this confusion, we write $(\psi(\cdot+y))\widehat{\ }(\xi)$, which means 'the Fourier transform of $x \mapsto \psi(x+y)$ evaluated at ξ'.

Question 5

For each $t \in \mathbf{R}$, consider the operators A_t and A_t^* defined on $f \in \mathcal{S}(\mathbf{R})$ by

$$A_t f(x) = f'(x) + txf(x) \quad \text{and} \quad A_t^* f(x) = -f'(x) + txf(x).$$

When $t = 1$ the operators A_t and A_t^* are sometimes called the *annihilation* and *creation* operators, respectively. Prove that for all $f, g \in \mathcal{S}(\mathbf{R})$ and

$$(f,g) = \int_{\mathbf{R}} f(x)\overline{g(x)}\,dx$$

we have

(a) $(A_t^* f, g) = (f, A_t g)$,
(b) $(A_t^* A_t f, f) \geq 0$,
(c) $A_t^* A_t = L_t - tI$, where $L_t(f)(x) = -f''(x) + t^2x^2f(x)$ and I is the identity operator.

Use these facts to prove $(L_t(f), f) \geq t(f,f)$ for all $f \in \mathcal{S}(\mathbf{R})$ and use it to provide an alternative proof of the Heisenberg uncertainty principle.

Solution.

(a) Using integration by parts and that complex conjugation and differentiation commute, we have

$$\begin{aligned}(A_t^* f, g) &= \int_{-\infty}^{\infty} (-f'(x) + txf(x))\overline{g(x)}dx = -\int_{-\infty}^{\infty} f'(x)\overline{g(x)}dx + \int_{-\infty}^{\infty} f(x)\overline{txg(x)}dx \\ &= f(x)g(x)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(x)\overline{g'(x)}dx + \int_{-\infty}^{\infty} f(x)\overline{txg(x)}dx = (f, A_t g)\end{aligned}$$

(b) Using (a), we have that

$$(A_t^* A_t f, g) = (A_t f, A_t g) = \|A_t f\|^2_{L^2(\mathbf{R})} \geq 0.$$

(c) We compute

$$\begin{aligned}A_t^* A_t f &= A_t^*(f'(\cdot) + t\cdot f(\cdot))(x) = -(f'(\cdot) + t\cdot f(\cdot))'(x) + tx(f'(\cdot) + t\cdot f(\cdot))(x) \\ &= -f''(x) - tf(x) - txf'(x) + tx(f'(x) + txf(x)) \\ &= -f''(x) - tf(x) + (tx)^2 f(x) = (L_t - tI)(f)(x),\end{aligned}$$

as required.

Now, using (c) and (b),

$$(L_t(f), f) = (A_t^* A_t(f) + tf, f) = (A_t^* A_t f, f) + t(f,f) \geq t(f,f).$$

This means that

$$\begin{aligned}0 &\leq (L_t(f), f) - t(f,f) = (-f'' + t^2(\cdot)^2 f - tf, f) \\ &= \int_{-\infty}^{\infty} (-f''(x) + t^2x^2f(x) - tf(x))\overline{f(x)}dx = \int_{-\infty}^{\infty} (|f'(x)|^2 + t^2|xf(x)|^2 - t|f(x)|^2)dx \qquad (5) \\ &= t^2\|\cdot f(\cdot)\|^2_{L^2(\mathbf{R})} - t\|f\|^2_{L^2(\mathbf{R})} + \|f'\|^2_{L^2(\mathbf{R})}.\end{aligned}$$

Thus, this quadratic polynomial in t has at most one real root, so consequently the discriminant is non-positive. Assuming $\|f\|_{L^2(\mathbf{R})} = 1$, we find that

$$1 - 4\|f'\|^2_{L^2(\mathbf{R})}\|\cdot f(\cdot)\|^2_{L^2(\mathbf{R})} \leq 0.$$

Therefore, using Plancherel's Theorem and standard properties of the Fourier transform,

$$
\begin{aligned}
1 &\le \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} |f'(x)|^2\, dx\right) \\
&\le \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} 16\pi^2 \xi^2 |\widehat{f}(\xi)|^2\, d\xi\right),
\end{aligned}
$$

which is the Heisenberg uncertainty principle.

Mark Scheme. Each of (a)-(c) was worth two points. Proving $(L_t(f), f) \ge t(f, f)$ was two points, as was proving the Heisenberg uncertainty principle.

Notes.

1. Several people don't seem to quite understand the statement of integration by parts on the interval $(-\infty, \infty) = \mathbf{R}$. The integral $\int_{\mathbf{R}} f(x)dx$, which is notation for
$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \quad (\text{when } f \in \mathcal{M}(\mathbf{R})),$$
seemed to confuse some people. We needed the identity
$$\int_{-\infty}^{\infty} f'(x)g(x)dx = f(x)g(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)g'(x)dx \tag{6}$$
When translating this into the above notation, some people wrote
$$\cancel{\int_{\mathbf{R}} f'(x)g(x)dx = f(x)g(x)\Big|_{\mathbf{R}} - \int_{\mathbf{R}} f(x)g'(x)dx,}$$
which I've crossed out because it is not clear what $f(x)g(x)\Big|_{\mathbf{R}}$ means and nobody defined it in any way. What should be written is either (6) or
$$\int_{\mathbf{R}} f'(x)g(x)dx = f(x)g(x)\Big|_{-\infty}^{\infty} - \int_{\mathbf{R}} f(x)g'(x)dx,$$
where
$$f(x)g(x)\Big|_{-\infty}^{\infty} := \lim_{R\to\infty} f(x)g(x)\Big|_{-R}^{R}$$
2. Instead of using the discriminant of the quadradic polynomial to prove the Heisenberg uncertainty principle some people tried to use the Cauchy-Schwarz inequality. However, they tried to apply it with a sum on the left-hand side which is clearly not the form of the inequality. Always be careful to understand what you are writing.