Chapter 1

Motivation - A Vibrating String

1.1 Lecture 1

1.1.1 Simple Harmonic Motion

In order to motivate our subsequent mathematical work, we wish to study a simple physical situation. Later we will attempt to describe a taut string fixed at both ends, for example, as in a musical instrument, but to lead on to this we first consider a how a weight moves when attached to a spring. To justify the mathematical model we first appeal to our physical intuition.

Imagine a spring fixed to a wall at one end with a mass attatched at the other. Suppose the floor on which it rests is frictionless. We want to derive an equation of motion for this system.

Assumption 1.1.1. *Hooke's Law The force $F = F(t)$ exerted by a spring is proportional to the length by which it is extended from its natural length. Namely, if we denote by $y(t)$ the displacement from this natural length at time t, we have*

$$F(t) = -ky(t),$$

where $k > 0$ is called the spring constant.

Assumption 1.1.2. *Newtons Second Law The net force exerted on a body is proportional to the acceleration of that body. Moreover, the constant of proportionality is the mass m of the body. Thus, $F(t) = my''(t)$*

Putting the above assumptions together we obtain

$$y''(t) = -\frac{k}{m}y(t)$$

or

$$y''(t) + c^2 y(t) = 0, \tag{1.1}$$

where $c = \sqrt{k/m}$. We recall that the solutions of this second order linear ordinary differential equation are

$$\begin{aligned} y(t) &= a\cos(ct) + b\sin(ct) \quad (\text{for } a, b \in \mathbf{C}) \\ &= A\cos(ct - \phi) \quad (\text{for } A \in \mathbf{C},\ \phi \in \mathbf{R}). \end{aligned}$$

This sinusoidal motion is called *simple harmonic motion*.

1.1.2 Derivation of the Wave Equation

We consider a taut string taut along an axis fixed at distance $x = 0$ and $x = 1$. To build a mathematical model of the string, we attempt to approximate it as a chain of N 'molecules'. The nth molecule is located at $x_n = \frac{nL}{N}$ and is of mass ρh, where $x_n - x_{n-1} = \frac{L}{N} = h$ and ρ is a given constant (which should be thought of as being the density of the string). Each molecule is free to move perpendicular

to the x-axis, but is attached to the two adjacent molecules by springs each with spring constant τ/h, where τ is a given constant (known as the coefficient of tension).

The perpendicular distance of the nth molecule from the x-axis at time t is denoted by $y_n = u(x_n, t)$. Newton's Second Law says the force on the nth molecule is

$$F(t) = \rho h \frac{\partial^2 u}{\partial t^2}(x_n, t).$$

Hooke's Law tells us that the force on the nth molecule from the $(n-1)$th is

$$\frac{\tau(y_{n-1} - y_n)}{h}.$$

Similarly, the force on the nth molecule from the $(n+1)$th is

$$\frac{\tau(y_{n+1} - y_n)}{h}.$$

Putting this together we find

$$\rho h \frac{\partial^2 u}{\partial t^2}(x_n, t) = \frac{\tau}{h}(y_{n-1} - 2y_n + y_{n+1})$$

and so,

$$\rho \frac{\partial^2 u}{\partial t^2}(x_n, t) = \frac{\tau}{h^2}(u(x_{n-1}, t) - 2u(x_n, t) + u(x_{n+1}, t)). \tag{1.2}$$

Now, for a twice differentiable function $F\colon \mathbf{R} \to \mathbf{C}$,

$$\frac{F(x-h) - 2F(x) + F(x+h)}{h^2} \to F''(x)$$

as $h \to 0$. Therefore, taking the limit $h \to 0$ in (1.2), we obtain

$$\rho \frac{\partial^2 u}{\partial t^2}(x, t) = \tau \frac{\partial^2 u}{\partial x^2}(x, t). \tag{1.3}$$

The equation (1.3) is called the wave equation.

1.2 Lecture 2

1.2.1 Standing and Travelling Waves

Standing Waves

A function $u\colon \mathbf{R} \times [0, \infty) \to \mathbf{C}$ is called a *standing wave* if it is of the form

$$u(x, t) = \phi(x)\psi(t),$$

for some functions $\phi\colon \mathbf{R} \to \mathbf{C}$ and $\psi\colon [0, \infty) \to \mathbf{C}$. These are 'waves' whose profile is scaled in height as time passes, but otherwise remains fixed.

Travelling Waves

A function $u\colon \mathbf{R} \times [0, \infty) \to \mathbf{C}$ is called a *travelling wave* if it is of the form

$$u(x, t) = F(x - ct)$$

for some function $F\colon \mathbf{R} \to \mathbf{C}$ and constant $c \in \mathbf{R}$. These are 'waves' whose profile is translated at speed c along the x-axis, but otherwise remains fixed.

1.2.2 Solving the wave equation: separation of variables

We wish to solve the wave equation. More precisely, given functions $f\colon \mathbf{R} \to \mathbf{C}$ and $g\colon \mathbf{R} \to \mathbf{C}$, we wish to find $u\colon \mathbf{R} \times [0,\infty) \to \mathbf{C}$ such that

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{1.4}$$

subject to the initial conditions

$$u(x,0) = f(x) \quad \text{and} \tag{1.5}$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad \text{for all } x \in \mathbf{R},$$

and the boundary conditions $u(0,t) = u(\pi,t) = 0$ for all $t \geq 0$. The physical picture we may wish to keep in mind is that u is the height of a string of length π, fixed at both ends $x = 0$ and $x = \pi$, with initial displacement f and initial velocity g.

We will look for solutions for u which are of the form

$$u(x,t) = \phi(x)\psi(t).$$

This method of finding solutions is called separation of variables. Under this assumption (1.4) becomes

$$\phi(x)\psi''(t) = \phi''(x)\psi(t).$$

Ignoring issues of dividing by zero, we can rewrite this as

$$\frac{\psi''(t)}{\psi(t)} = \frac{\phi''(x)}{\phi(x)} \tag{1.6}$$

Since the left-hand side of (1.6) depends only on t and the right-hand side of (1.6) depends only on x, we can conclude each side is equal to a constant λ. Thus we obtain

$$\psi''(t) - \lambda\psi(t) = 0 \tag{1.7}$$
$$\phi''(x) - \lambda\phi(x) = 0 \tag{1.8}$$

These equations are of the form (1.1) with $\lambda = -c^2$ and so all the possible solutions are

$$\psi(t) = A\cos(ct) + B\sin(ct) \quad (A, B \in \mathbf{C})$$
$$\phi(x) = \tilde{A}\cos(cx) + \tilde{B}\sin(cx) \quad (\tilde{A}, \tilde{B} \in \mathbf{C})$$

but as $u(0,t) = 0$, we require $\psi(x) = \tilde{B}\sin(cx)$, that is $\tilde{A} = 0$. Also, if we assume $\tilde{B} \neq 0$, which will be the case if we are to find a non-zero solution u, as $u(\pi,t) = 0$, we require $0 = \phi(\pi) = \tilde{B}\sin(c\pi)$ and so c must be an integer. Thus our solution is of the form

$$u(x,t) = (A\cos(ct) + B\sin(ct))\sin(cx)$$

for $c \in \mathbf{Z}$ and $A, B \in \mathbf{C}$.

Since (1.4) is linear we can add various solutions together to get a new solution, therefore our guess at a general solution to (1.4) is

$$u(x,t) = \sum_{m=1}^{\infty} (A_m\cos(mt) + B_m\sin(mt))\sin(mx), \tag{1.9}$$

bearing in mind that we can absorb the sum over negative integers into the sum over the positive. Combining (1.9) with (1.5), given a function $f\colon [0,\pi] \to \mathbf{C}$ such that $f(0) = f(\pi) = 0$, we need to find coefficients $\{A_m\}_m$ such that

$$f(x) = \sum_{m=1}^{\infty} A_m \sin(mx). \tag{1.10}$$

We can easily guess a candidate for A_m. We compute

$$\int_0^\pi f(x)\sin(nx)dx = \int_0^\pi \left(\sum_{m=1}^{\infty} A_m\sin(mx)\right)\sin(nx)dx$$
$$= \sum_{m=1}^{\infty} A_m \int_0^\pi \sin(mx)\sin(nx)dx.$$

As the integral in the sum is zero when $n \neq m$ and equals $\frac{\pi}{2}$ when $n = m$, the sum has just one non-zero term. We find that

$$\int_0^\pi f(x)\sin(nx)dx = A_n\frac{\pi}{2},$$

that is,

$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)dx. \tag{1.11}$$

The right-hand side of (1.11) is called the nth *Fourier sine coefficient*.

If we suppose that we could write a function $f\colon [0,\pi] \to \mathbf{C}$ as a series (1.10), we observe that the formula also makes sense on say $[-\pi,\pi]$. We notice however, that this would produce an odd function. Given that an arbitrary function on $[-\pi,\pi]$ is the sum of an odd and an even function, for a function $f\colon [-\pi,\pi] \to \mathbf{C}$ we might try to look for a representation of the form

$$f(x) = \sum_{n=0}^{\infty} A_n\sin(nx) + B_n\cos(nx)$$

or equally, using Euler's identity,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \tag{1.12}$$

In the following lectures we will discover what we need to assume about a function f in order to write it as (1.12) and the properties of any such coefficients $\{c_n\}_n$.

Remark 1.2.1. *In the last two lectures we have used physical arguments and rough calculations to motivate asking questions about how a function may be represented. We have been careless in performing certain steps, such as dividing by quantities which may be zero and interchanging integrals and infinite sums. Now we have these questions in mind, we want to set about the task of making rigorous mathematical statements. So it is only now that our real work begins and one must bear in mind that we have not yet proved anything.*

Chapter 2

Fourier Series

2.1 Lecture 3

Given an integrable function $f : [-\pi, \pi] \to \mathbb{C}$ we define a sequence $\{a_n\}_{n\in\mathbf{Z}}$ as

$$a_n = \hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

for each $n \in \mathbf{Z}$. The numbers $a_n = \hat{f}(n)$ are called the Fourier coefficients of our function f. Observe that the assumption that f is integrable means that Fourier coefficients are well-defined.

Here we are defining Fourier coefficients for functions on $[-\pi, \pi]$, but we can replace the interval $[-\pi, \pi]$ with any bounded inteval provided we introduce appropriate constants. We leave this as an exercise for the interested reader. In particular, the interval $[-1, 1]$ is a common choice in the literature.

We can identify 2π-periodic functions on $\mathbf{R}$ with functions on $[-\pi, \pi]$ such that $f(-\pi) = f(\pi)$ by simply restricting the domain of the function. With this identification in mind, when we say $f : [-\pi, \pi] \to \mathbf{C}$ is continuous provided $f(-\pi) = f(\pi)$ and the 2π-periodic extension $g : [-\pi, \pi] \to \mathbf{C}$ defined as

$$g(x) = f(x + 2k\pi)$$

for $x \in \mathbf{R}$ and k such that $x+2k\pi \in [-\pi, \pi]$ is continuous in the usual sense. We define differentiability for $f : [-\pi, \pi] \to \mathbf{C}$ similarly.

The Fourier series of a function f is defined to be

$$x \mapsto \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}.$$

At this point we just consider the series as a formal series and do not worry about issues of convergence. Questions regarding whether or not such a series converges and to what the series converges will occupy us for a large part of the course. However, it is worth noting at this point that we do not yet know the Fourier series of f will converge to f in any sense. Although we will soon discover assumptions under which this is the case, we will also see situations where such a convergence fails.

In order to address issues of convergence, we wish to considered the partial sums of the Fourier series of a function f. The Nth *partial sum* of the Fourier series of f is defined to be

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$$

Example 2.1.1. *Compute the Fourier series of $f : [-\pi, \pi] \to \mathbf{C}$ defined by*

$$f(\theta) = \theta \quad \textit{for all } \theta \in [-\pi, \pi].$$

Solution. If $n \neq 0$ then

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}d\theta \\
&= \frac{1}{2\pi}\left(\left[\frac{-\theta e^{in\theta}}{in}\right]_{-\pi}^{\pi} + \frac{1}{2\pi in}\int_{-\pi}^{\pi} e^{-in\theta}d\theta\right) \\
&= \frac{1}{2\pi}\left(\left[\frac{-\pi}{in}(-1)^n - \frac{\pi}{in}(-1)^n\right] + \frac{1}{2\pi in}\left[\frac{-e^{-in\theta}}{in}\right]_{-\pi}^{\pi}\right) \\
&= \frac{1}{2\pi}\left(\frac{-2\pi}{in}(-1^n)\right) = \frac{(-1)^{n+1}}{in}
\end{aligned}$$

and

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta d\theta = 0.$$

Therefore the Fourier series of f is

$$\theta \mapsto \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n+1}}{in}e^{in\theta}.$$

Formally, we can rearrange this as

$$\theta \mapsto 2\sum_{n=1}^{\infty} (-1)^{n+1}\frac{\sin(n\theta)}{n}.$$

□

Example 2.1.2. *Consider the trignometric polynomial defined by*

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

This is called the N^{th} Dirichlet kernel. Observe that

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)D_N(x-y)dy &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\sum_{n=-N}^{N} e^{in(x-y)}dy \\
&= \sum_{n=-N}^{N} \frac{e^{inx}}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}dy \\
&= \sum_{n=-N}^{N} \hat{f}(n)e^{inx} = S_N(f)(x).
\end{aligned}$$

We will compute a formula for D_N in the next homework.

2.2 Lecture 4

2.2.1 Uniqueness of Fourier Series

Theorem 2.2.1. *Suppose that f is a bounded integrable function on $[-\pi, \pi]$. If $\hat{f}(n) = 0$ for all $n \in \mathbf{Z}$ and f is continuous at a point x_0, then $f(x_0) = 0$.*

Proof. We will assume the conclusion of the Theorem is false, that is $f(x_0) \neq 0$, and aim to obtain a cotradiction. By considering, if necessary, $-f$ in place of f we can assume $f(x_0) > 0$.

We will first seek a contradition in the special case that $x_0 = 0$ and f is real-valued. We observe that since $\hat{f}(n) = 0$,

$$\int_{-\pi}^{\pi} f(x)P(x)dx = 0, \tag{2.1}$$

where $P(x)$ is a trignometric polynomial, that is

$$P(x) = \sum_{n\in\mathbf{Z}} c_n e^{inx},$$

where $c_n \in \mathbf{C}$. We will now construct a trigonometric polynomial such that (2.1) does not hold, and so obtain a contradiction.

Since f is continuous at 0 choose $0 < \delta < \frac{\pi}{2}$ such that

$$f(x) > \frac{f(0)}{2} \text{ for } |x| < \delta$$

Let $p(x) = \varepsilon + \cos(x)$ where $\varepsilon > 0$ is so small that $|p(x)| < 1 - \frac{\varepsilon}{2}$ whenever $\delta < |x| < \pi$. Now we choose $0 < \nu < \delta$, so that $p(x) \geq 1 + \frac{\varepsilon}{2}$ for $|x| < \nu$. Finally, let $p_k(x) = (p(x))^k$ and select B so that $|f(x)| \leq B$ for all x.

Since each p_k is a trignometric polynomial, and $\hat{f}(n) = 0$ we have

$$\int_{-\pi}^{\pi} f(x)p_k(x) = 0,$$

for all $k \in \mathbf{N}$. However

$$\left|\int_{\delta\leq|x|} f(x)p_k(x)dx\right| \leq 2\pi B(1 - \frac{\epsilon}{2})^k \to 0 \text{ as } k \to \infty$$

and since $p(x), f(x) \geq 0$ whenever $|x| < \delta$ then

$$\int_{\nu<|x|<\delta} f(x)p_k(x)dx \geq 0.$$

Finally,

$$\int_{|x|\leq\nu} f(x)p_k(x)dx \geq 2\nu\frac{f(0)}{2}(1 + \frac{\epsilon}{2})^k \to \infty \quad \text{as } k \to \infty$$

so

$$\int_{-\pi}^{\pi} f(x)p_k(x)dx \to \infty \text{ as } k \to \infty$$

Which is a contradiction.

Now to deal with complex valued functions, set $u(x) + iv(x)$ and if we compute the Fourier coefficients of u and v we see that they are also zero and we may apply the argument above to (if necessary, the negative of) the real and imaginary parts separately.

Finally, for $x_0 \neq 0$ we apply the argument to a translation of f. □

Now let us restate this result for functions which are continuous at all points in their domain.

Corollary 2.2.2. *If f is continuous on $[-\pi, \pi]$ (and $f(-\pi) = f(\pi)$) and $\hat{f}(n) = 0$ for all $n \in \mathbf{N}$ then $f \equiv 0$*

Corollary 2.2.3. *The map which takes continuous functions to their Fourier series is injective. That is to say, if two continuous functions f and g on $[-\pi, \pi]$ are such that $\hat{f}(n) = \hat{g}(n)$ for all $n \in \mathbf{N}$, then $f \equiv g$.*

Proof. Apply Corollary 2.2.2 to $f - g$. □

2.3 Lecture 5

Corollary 2.3.1. *Suppose that $f\colon [-\pi, \pi] \to \mathbf{C}$ is a continuous function and the Fourier series of f converges absolutely (that is, $\sum_n |\hat{f}(n)|$ converges). Then, the Fourier series converges uniformly to f, that is*

$$\lim_{N\to\infty} S_N(f)(x) = f(x) \quad \textit{uniformly in } x.$$

Proof. Define

$$g(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$$

and observe that

$$|S_N(f)(x) - g(x)| = \left|\sum_{|n|>N} \hat{f}(n)e^{inx}\right| \leq \sum_{|n|>N} |\hat{f}(n)| \to 0 \text{ as } N \to \infty$$

uniformly in x. That is, $S_N(f)$ converges uniformly to g. Since $S_N(f)$ is continuous and converges uniformly to g, g is continuous. Finally, since the sum defining g is absolutely convergent, we can intechange integration and summation and so

$$\hat{g}(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{inx}\right)e^{-imx}dx = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(n)e^{i(n-m)x}dx = \hat{f}(m),$$

for all $m \in \mathbf{Z}$. Therefore we can apply Corollary 2.2.3 to conclude $f \equiv g$ □

Given Corollary 2.3.1, a natural question to ask is, 'What conditions on f would guarantee the absolute convergence of it's Fourier series?'

First we fix some notation: We say $\hat{f}(n) = O(\frac{1}{n^2})$ as $|n| \to \infty$ to mean $|\hat{f}(n)| \leq C\frac{1}{n^2}$ for large $|n|$.

Corollary 2.3.2. *Suppose f is a twice continuously differentiable function on the unit circle. Then $\hat{f}(n) = O(\frac{1}{n^2})$ as $|n| \to \infty$.*

Proof. We have, for $n \neq 0$,

$$\begin{aligned}
\hat{f}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\
&= \frac{1}{2\pi}\left[\frac{-1}{in}f(x)e^{-inx}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f'(x)e^{-inx}}{-in}dx \\
&= \frac{1}{2\pi in}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx \quad \left(= \frac{1}{in}\hat{f'}(n)\right) \\
&= \frac{1}{2\pi in}\left(\left[\frac{f'(x)e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f''(x)\frac{e^{-inx}}{-in}dx\right) \\
&= \frac{-1}{2\pi n^2}\int_{-\pi}^{\pi} f''(x)e^{-inx}dx.
\end{aligned}$$

Therefore,

$$|\hat{f}(n)| = |\frac{1}{2\pi n^2}\int_{-\pi}^{\pi} f''(x)e^{-inx}dx| \leq \frac{C}{n^2}.$$

□

Remark 2.3.3. *(i) The proof above shows that if f is continuously differentiable then*

$$\widehat{f'}(n) = in\hat{f}(n).$$

Therefore by induction

$$\widehat{f^{(k)}}(n) = (in)^k \hat{f}(n),$$

for a function f that is k-times continuously differentiable.
(ii) The hypothesis of the above corrollary can actually be weakened to f satisfying a Hölder continuity condition: $\sup_x |f(x+t) - f(x)| \leq A|t|^\alpha$ for some $\alpha \in (0, 1/2)$ and all t. We will not discuss the proof of this here.

2.4 Lecture 6

2.4.1 Mean-Square Convergence of Fourier Series

We first recall some facts about inner product spaces. An inner product space over **C** is a vector space V over **C** together with a positive definite, bilinear from

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}.$$

Being positive definite means

$$\langle v, v \rangle \geq 0 \text{ for all } v \in V, \text{ and } \langle v, v \rangle = 0 \text{ if and only if } v = 0.$$

Bilinearity means

$$\langle av + bw, u \rangle = a\langle v, u \rangle + b\langle w, u \rangle \quad \text{and}$$
$$\langle u, av + bw \rangle = \bar{a}\langle u, v \rangle + \bar{b}\langle u, w \rangle$$

for all $u, v, w \in V$ and $a, b \in \mathbf{C}$. An inner product space is said to be Hermitian if

$$\langle v, w \rangle = \overline{\langle w, v \rangle}$$

for all $v, w \in V$. We can define a norm $\| \cdot \|$ on the inner product space $(V, \langle \cdot, \cdot \rangle)$ as $\|v\| = \langle v, v \rangle$. We say v and w are orthogonal if $\langle v, w \rangle = 0$. Pythagoras' theorem states that if v and w are orthogonal, then

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2.$$

The Cauchy-Schwarz Inequality states that for any $v, w \in V$ we have

$$|\langle v, w \rangle| \leq \|v\| \|w\|$$

and that

$$\|v + w\| \leq \|v\| + \|w\|$$

is called the Triangle Inequality.

Example 2.4.1. *The vector space $l^2(\mathbf{Z})$ over **C** is the set of all two-sided sequences of complex numbers, that is*

$$\{a_n\}_{n \in \mathbb{Z}} = \{\dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots\}$$

such that

$$\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty.$$

Addition and scalar multiplication in $l^2(\mathbb{Z})$ are defined component wise, that is

$$\{a_n\}_n + \{b_n\}_n = \{a_n + b_n\}_n \quad \text{and} \quad \lambda\{a_n\}_n = \{\lambda a_n\}_n.$$

We define the inner product of two sequences, $A = \{a_n\}_n$, $B = \{b_n\}_n$ to be

$$\langle A, B \rangle_{l^2(\mathbf{Z})} = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n.$$

Therefore the norm of $l^2(\mathbf{Z})$ is

$$\|A\|_{l^2(\mathbf{Z})} = \left(\sum_{n=-\infty}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}.$$

Cauchy sequences in $l^2(\mathbb{Z})$ converge to elements in $l^2(\mathbb{Z})$, that is $l^2(\mathbf{Z})$ is a complete inner product space and is thus a Hilbert space.

Example 2.4.2. *Consider the set of continuous complex-valued functions f on the circle (or equivalently on $[-\pi, \pi]$) such that*

$$\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta < \infty.$$

We define addition and scalar multiplication for this inner product space pointwise, that is

$$(f + g)(\theta) = f(\theta) + g(\theta) \quad \text{and} \quad (\lambda f)(\theta) = \lambda f(\theta),$$

and define an inner product as

$$\langle f, g \rangle_{L^2(\mathbf{T})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)} d\theta.$$

So the norm corresponding to this inner product is

$$\|f\|_{L^2(\mathbf{T})} = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta \right)^{\frac{1}{2}}.$$

*Note that it is not too hard to find a sequence of continuous functions $\{f_n\}_n$ which is a Cauchy sequence in the $L^2(\mathbf{T})$-norm, but whose pointwise limit $\lim_{n\to\infty} f_n$ is not continuous. Thus the space is not complete. However, one may take the completion, using equivalence classes of Cauchy sequences, in order to obtain a Hilbert space. We denote this Hilbert space by $L^2(\mathbf{T})$. For our purposes, we will be able to check a function f is in $L^2(\mathbf{T})$ if one can find a sequence of continuous functions on **T** which is a Cauchy sequence in the $L^2(\mathbf{T})$-norm and has the pointwise limit f.*

Theorem 2.4.3 (Mean-Square Convergence)**.** *If $f \in L^2(\mathbf{T})$, then*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta) - S_N(f)|^2 d\theta \to 0$$

as $N \to \infty$.

To prove this we need a lemma.

Lemma 2.4.4 (Best Approximation)**.** *If $f \in L^2(\mathbf{T})$, then, for each $N \in \mathbf{N}$ and any $c_n \in \mathbf{C}$ ($n = -N, \dots, N$),*

$$\|f - S_N(f)\|_{L^2(\mathbf{T})} \leq \|f - \sum_{n=-N}^{N} c_n e_n\|_{L^2(\mathbf{T})}$$

where $e_n(x) = e^{inx}$. Moreover equality holds precisely when $c_n = \hat{f}(n)$ for each $n = -N, \dots, N$.

This basically says that $\hat{f}(n)$ gives the best approximation of f by trigonometric polynomials of order N in L^2.

Proof. In our new notation

$$\hat{f}(n) = \langle f, e_n \rangle_{L^2(\mathbf{T})}$$

and

$$\langle e_n, e_m \rangle_{L^2(\mathbf{T})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\theta(n-m)} d\theta = \begin{cases} 1, & n = m; \\ 0, & n \neq m. \end{cases}$$

Consider
$$f - \sum_{n=-N}^{N} c_n e_n = f - S_N(f) + \sum_{n=-N}^{N} b_n e_n$$
where $b_n = \hat{f}(n) - c_n$. Now
$$\begin{aligned}\langle f - S_N(f), \sum_{n=-N}^{N} b_n e_n\rangle &= \langle f, \sum_{n=-N}^{N} b_n e_n\rangle - \langle S_N(f), \sum_{n=-N}^{N} b_n e_n\rangle \\ &= \sum_{n=-N}^{N} \bar{b}_n \langle f, e_n\rangle - \sum_{m=-N}^{N} \hat{f}(m)\bar{b}_n \langle e_m, e_n\rangle \\ &= \sum_{n=-N}^{N} \hat{f}(n)\bar{b}_n - \sum_{n=-N}^{N} \hat{f}(n)\bar{b}_n \\ &= 0\end{aligned}$$
So, by Pythagoras' Theorem we have that
$$\|f - \sum_{n=-N}^{N} c_n e_n\|^2_{L^2(\mathbf{T})} = \|f - S_N(f)\|^2_{L^2(\mathbf{T})} + \underbrace{\|\sum_{n=-N}^{N} b_n e_n\|^2_{L^2(\mathbf{T})}}_{\geq 0} \geq \|f - S_N(f)\|^2_{L^2(\mathbf{T})},$$
as claimed. □

2.5 Lecture 7

Proof of Theorem 2.4.3. First let us suppose that f is continuous. We can approximate a continuous function uniformly by a trigonometric polynomial, so there exists a trigonometric polynomial P such that
$$|f(\theta) - P(\theta)| < \varepsilon$$
for each fixed $\epsilon > 0$.
P will have some degree, say M (possibly very large). So, by our lemma we know that
$$\|f - S_N(f)\|_{L^2(\mathbf{T})} \leq \|f - P\|_{L^2(\mathbf{T})}$$
for $N \geq M$. But
$$\|f - P\|_{L^2(\mathbf{T})} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta) - P(\theta)|^2 d\theta\right)^{\frac{1}{2}} < \varepsilon.$$
This proves the theorem for continuous f.

Now suppose f is merely in $L^2(\mathbf{T})$ and fix $\varepsilon > 0$. Now we can find a continuous function g which approximates f in the $L^2(\mathbf{T})$-norm, so we have that
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta) - g(\theta)|^2 d\theta < \varepsilon^2.$$
Then choose a trigonometric polynomial P of some order M such that $|g(\theta) - P(\theta)| < \varepsilon$ so
$$\|f - S_N(f)\|_{L^2(\mathbf{T})} \leq \|f(\theta) - P(\theta)\|_{L^2(\mathbf{T})} \leq \|f - g\|_{L^2(\mathbf{T})} + \|g - P\|_{L^2(\mathbf{T})} < 2\varepsilon$$
for $N \geq M$. □

Theorem 2.5.1 (Parseval's Identity)**.** *Let $f \in L^2(\mathbf{T})$, then $\left\{\hat{f}(n)\right\}_{n\in\mathbf{Z}} \in l^2(\mathbf{Z})$ and*
$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta\right)^{\frac{1}{2}} = \|f\|_{L^2(\mathbf{T})} = \|\left\{\hat{f}(n)\right\}_{n\in\mathbb{Z}}\|_{l^2(\mathbf{Z})} = \left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\right)^{\frac{1}{2}}.$$

Proof. We have
$$f = (f - S_N(f)) + S_N(f).$$
We have seen in the proof of Lemma 2.4.4 that $f - S_N(f)$ is orthogonal to $S_N(f)$ (to see this take $c_n = 0$) and so
$$\|f\|_{L^2(\mathbf{T})} = \|f - S_N(f)\|^2_{L^2(\mathbf{T})} + \|S_N(f)\|^2_{L^2(\mathbf{T})}.$$
So then
$$\begin{aligned}\|S_N(f)\|^2_{L^2(\mathbf{T})} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{|n|\leq N} \hat{f}(n) e_n\right)\left(\sum_{|m|\leq N} \overline{\hat{f}(m)} e_{-m}\right) \\ &= \frac{1}{2\pi}\sum_{n,m} \hat{f}(n)\overline{\hat{f}(m)}\langle e_n, e_{-m}\rangle \\ &= \sum_{|n|\leq N} |\hat{f}(n)|^2.\end{aligned}$$
Thus
$$\|f\|^2_{L^2} = \|f - S_N(f)\|^2_{L^2} + \sum_{|n|\leq N} |\hat{f}(n)|^2$$
and taking the limit as $N \to \infty$ we find that
$$\|f\|^2_{L^2(\mathbf{T})} = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \tag{2.2}$$
by Theorem 2.4.3. □

Remark 2.5.2. *In fact we can generalise* (2.2) *to prove that for any orthonormal family of functions $\{l_n\}_{n\in\mathbf{Z}}$ with $a_n := \langle f, l_n\rangle$ we have that*
$$\sum_{n=-N}^{N} |a_n|^2 \leq \|f\|^2_{L^2(\mathbf{T})}.$$
This is called Bessel's Inequality. (In addition to this, equality holds precisely when $\{l_n\}_{n\in\mathbf{Z}}$ is a basis for $L^2(\mathbf{T})$.)

Corollary 2.5.3. *Suppose $f, g \in L^2(\mathbf{T})$ then*
$$\langle f, g\rangle_{L^2(\mathbf{T})} := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)} d\theta = \sum_{-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)}.$$

Proof. Use the Polarisation Identity and Parseval's Identity. □

Chapter 3

The Fourier Transform

3.1 Lecture 8

3.1.1 Basic Properties

Definition 3.1.1. *A function $f\colon \mathbf{R} \to \mathbf{C}$ is said to be of moderate decrease if f is continuous and there exists $A > 0$ such that*

$$|f(x)| \leq \frac{A}{1+|x|^2}$$

for all $x \in \mathbf{R}$. We denote the set of all such f by $\mathcal{M}(\mathbf{R})$. Observe that $\mathcal{M}(\mathbf{R})$ is a vector space with respect to the usual notions of addition and scalar mulitiplication of functions.

We can define:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-n}^{n} f(x)dx$$

for $f \in \mathcal{M}(\mathbf{R})$

Proposition 3.1.2. *Let $f, g \in \mathcal{M}(\mathbf{R})$. Then*

$$\int_{\mathbf{R}} (af(x)+bg(x))dx = a\int_{\mathbf{R}} f(x)dx + b\int_{\mathbf{R}} g(x)dx \quad (linearity)$$

for each $a, b \in \mathbf{C}$,

$$\int_{\mathbf{R}} f(x-h)dx = \int_{\mathbf{R}} f(x)dx \quad (translation\ invariance)$$

for each $h \in \mathbf{R}$,

$$\delta^{-1}\int f(\delta x)dx = \int f(x)dx \quad (scaling\ under\ dilation)$$

for each $\delta > 0$, and

$$\int |f(x-h)-f(x)|dx \to 0 \quad (continuity)$$

as $h \to 0$.

Definition 3.1.3. *If $f \in \mathcal{M}(\mathbf{R})$, we define the Fourier transform of f to be a function $\widehat{f}\colon \mathbf{R} \to \mathbf{C}$ given by*

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}dx$$

for all $\xi \in \mathbf{R}$.

Definition 3.1.4. *A function $f : \mathbf{R} \to \mathbf{C}$ is said to be a Schwarz function if, for each $l, k \in \mathbf{N}\cup\{0\}$ there exists a constant $c_{l,k}$ such that*

$$|x^k f^{(l)}(x)| \leq c_{l,k}.$$

We denote the set of all such f by $\mathcal{S}(\mathbf{R}) = \mathcal{S}$.

Example 3.1.5. *1. The function f defined as $f(x) = e^{-x^2}$ for all $x \in \mathbf{R}$ is a Schwarz function. This function is called a Gaussian and turns up in several different contexts.*

2. Functions which are smooth and zero outside a bounded interval are Schwarz functions. That is, functions f such that $f \in C^\infty$ and $\mathrm{supp} f := \overline{\{x|f(x) \neq 0\}} \subset [-R, R]$ for some $R > 0$.

3. The function g defined by $g(x) = e^{-|x|}$ for all $x \in \mathbf{R}$ is not a Schwarz function as, although it decays sufficiently quickly, g is not differentiable.

We will write $f(x) \overset{\widehat{}}{\mapsto} \hat{f}(\xi)$ to mean the Fourier transform of f is $\hat{f}$

Proposition 3.1.6. *If $f \in \mathcal{S}(\mathbf{R})$ then we have:*

(i) $f(x+h) \overset{\widehat{}}{\mapsto} e^{2\pi ih\xi}\hat{f}(\xi)$ for $h \in \mathbf{R}$;

(ii) $f(x)e^{-2\pi ixh} \overset{\widehat{}}{\mapsto} \hat{f}(\xi+h)$ for $h \in \mathbf{R}$;

(iii) $f(\delta x) \overset{\widehat{}}{\mapsto} \delta^{-1}\hat{f}(\delta^{-1}\xi)$ for $\delta > 0$;

(iv) $f'(x) \overset{\widehat{}}{\mapsto} 2\pi i\xi\hat{f}(\xi)$;

(v) $-2\pi ixf(x) \overset{\widehat{}}{\mapsto} \frac{d}{d\xi}(\hat{f})(\xi)$.

Proof. All of these results follow easily from the definition of the Fourier transform. We will only prove (iii) here. We have that

$$\begin{aligned}(f(\delta\cdot))\widehat{\ }(\xi) &= \int_{-\infty}^{\infty} f(\delta x)e^{-2\pi ix\xi}dx \\ &= \int_{-\infty}^{\infty} f(y)e^{-2\pi i\frac{y\xi}{\delta}}\frac{dy}{\delta} = \frac{1}{\delta}\widehat{f}(\frac{\xi}{\delta}),\end{aligned}$$

as claimed. □

Theorem 3.1.7. *If $f \in \mathcal{S}(\mathbf{R})$, then $\widehat{f} \in \mathcal{S}(\mathbf{R})$.*

Proof. We compute using Proposition 3.1.6 that

$$\begin{aligned}\left|\xi^k\left(\frac{d}{d\xi}\right)^l(\widehat{f})(\xi)\right| &= \left|\left[\frac{1}{(2\pi i)^k}((-2\pi i(\cdot))^l f(\cdot))^{(k)}\right]^{\widehat{}}(\xi)\right| \\ &= \left|\int_{-\infty}^{\infty}\frac{1}{(2\pi i)^k}\frac{d^k}{dx^k}((-2\pi ix)^l f(x))e^{-2\pi ix\xi}dx\right| \leq \int_{-\infty}^{\infty}\frac{C_{k,l}}{1+x^2} \leq A_{k,l} < \infty,\end{aligned}$$

which shows that $\widehat{f} \in \mathcal{S}(\mathbf{R})$. □

3.2 Lecture 9

Theorem 3.2.1. *If $f(x) = e^{-\pi x^2}$, then $\hat{f}(\xi) = e^{-\pi\xi^2}$*

Proof. This is left as an exercise (Homework 3). □

Remark 3.2.2. *Although we do not know a primative of $x \mapsto e^{-\pi x^2}$, we know that*

$$\int_{-\infty}^{\infty} e^{-\pi x^2}dx = 1.$$

To see this we write the square of the integral as a repeated integral and then change variables, so we find that

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-\pi x^2}dx\right)^2 &= \int_{-\infty}^{\infty} e^{-\pi x^2}\int_{-\infty}^{\infty} e^{-\pi y^2}dydx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)}dydx\\ &= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-\pi r^2}rdrd\theta\\ &= 2\pi\int_0^{\infty} re^{-\pi r^2}dr\\ &= 1\end{aligned}$$

Corollary 3.2.3. *If $\delta > 0$ and $K_\delta(x) = \delta^{\frac{-1}{2}}e^{-\pi\frac{x^2}{\delta}}$, then $\hat{K_\delta}(\xi) = e^{-\pi\delta\xi^2}$*

Proof. Apply (iii) from Proposition 3.1.6 (with δ replaced by $\delta^{\frac{1}{2}}$). □

Remark 3.2.4. *The family of kernels $\{K_\delta\}_\delta$ is what is called a family of good kernels. A collection of functions $\{k_\delta\}_\delta$ is called a family of good kernels if*

1. $\int_{-\infty}^{\infty} k_\delta(x)dx = 1$,
2. $\int_{-\infty}^{\infty} |k_\delta(x)|dx \le M$, *for some* $M > 0$, *and*
3. *for each* $\eta > 0$ *we have that* $\int_{|x|>\eta} |k_\delta(x)|dx \to 0$ *as* $\delta \to 0$

The first two properties are easily verified for $\{K_\delta\}_\delta$ and the third can be checked using Remark 3.2.2. Indeed,

$$\begin{aligned}\int_{|x|>\eta} |K_\delta(x)|dx &= \int_{|x|>\eta} \delta^{\frac{-1}{2}}e^{-\pi\frac{x^2}{\delta}}dx\\ &= \int_{|y|>\eta/\delta^{\frac{1}{2}}} e^{-\pi y^2}dy \to 0\end{aligned}$$

as $\delta \to 0$.

For two functions $f, g \in \mathcal{S}(\mathbf{R})$ we define their convolution $f * g$ to be

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt.$$

Theorem 3.2.5. *If $f \in \mathcal{S}(\mathbf{R})$, then*

$$(f * K_\delta)(x) = \int f(x-t)K_\delta(t)dt \to f(x)$$

uniformly in x as $\delta \to 0$.

Proof. First we claim f is uniformly continuous. Indeed, for a fixed $\varepsilon > 0$, there exists $R > 0$ such that $|f(x)| < \frac{\epsilon}{4}$ whenever $|x| > R$. Moreover, as f is continuous on the compact set $[-R, R]$, f is uniformly continuous on $[-R, R]$. Hence we can find $\eta > 0$ such that

$$|f(x) - f(y)| < \varepsilon/2 \quad \text{whenever} \quad |x - y| < \eta.$$

Now,

$$(f * K_\delta)(x) = \int_{-\infty}^{\infty} K_\delta(t)[f(x-t) - f(x)]dt,$$

so

$$|(f * K_\delta)(x) - f(x)| \le \int_{|t|>\eta} |K_\delta(t)[f(x-t) - f(x)]|dt + \int_{|t|\le\eta} |K_\delta(t)[f(x-t) - f(x)]|dt.$$

This first integral tends to zero as $\delta \to 0$ by property 3 in Remark 3.2.4 and the second integral is majorised by $\varepsilon\int|K_\delta|dt/2 = \varepsilon/2$. Consequently, the sum is less than ε for sufficiently small δ. □

3.3 Lecture 10

3.3.1 The Fourier Inversion Formula

Proposition 3.3.1. *If $f, g \in \mathcal{S}(\mathbf{R})$, then*

$$\int_{-\infty}^{\infty} f(x)\widehat{g}(x)dx = \int_{-\infty}^{\infty} \widehat{f}(y)g(y)dy$$

Proof. We have that

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\widehat{g}(x)dx &= \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} g(y)e^{-2\pi ixy}dydx\\ &= \int_{-\infty}^{\infty} g(y)\int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dxdy\\ &= \int_{-\infty}^{\infty} g(y)\widehat{f}(y)dy,\end{aligned}$$

as required. □

Theorem 3.3.2 (Fourier Inversion)**.** *If $f \in \mathcal{S}(\mathbf{R})$, then*

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi ix\xi}d\xi,$$

for all $x \in \mathbf{R}$.

Proof. First we claim that

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi)d\xi.$$

To see this set $G_\delta(x) = e^{-\pi\delta x^2}$, so

$$\widehat{G}(\xi) = \delta^{-\frac{1}{2}}e^{-\pi\frac{\xi^2}{\delta}} = K_\delta(\xi)$$

Then, by Proposition 3.3.1,

$$f * K_\delta(0) = \int_{-\infty}^{\infty} f(x)K_\delta(x)dx = \int_{-\infty}^{\infty} f(x)\widehat{G}_\delta(x)dx = \int_{-\infty}^{\infty} \widehat{f}(x)G_\delta(x)dx.$$

The left-hand side tends to $f(0)$ as $\delta \to 0$ by Theorem 3.2.5 and the right-hand side tends to $\int_{-\infty}^{\infty} \widehat{f}(x)dx$ as $\delta \to 0$. (The rigorous proof of the second limit is left as an exercise.)

In general (that is for $x \neq 0$) for a fixed x let $F(y) = f(x + y)$. Then

$$f(x) = F(0) = \int_{-\infty}^{\infty} \widehat{F}(\xi)d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{2\pi i\xi x}d\xi,$$

by (i) of Proposition 3.1.6. □

Remark 3.3.3. *The inversion formula is very similar to the Fourier transform itself. In fact it only differs by a minus sign. By defining*

$$\mathcal{F}(f)(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}dx$$

and

$$\mathcal{F}^*(g)(x) = \int_{-\infty}^{\infty} g(\xi)e^{2\pi ix\xi}d\xi$$

we have $\mathcal{F}^* \circ \mathcal{F} = I$ on $\mathcal{S}(\mathbf{R})$ and, since $\mathcal{F}(f)(\xi) = \mathcal{F}^*(f)(-\xi)$, $\mathcal{F} \circ \mathcal{F}^* = I$, where I is the identity operator on $\mathcal{S}(\mathbf{R})$.

Corollary 3.3.4. *The Fourier transform is a bijective mapping on $\mathcal{S}(\mathbf{R})$.*

3.3.2 Plancherel's Theorem

Proposition 3.3.5. *If $f, g \in \mathcal{S}(\mathbf{R})$ then*

(i) $f * g \in \mathcal{S}(\mathbf{R})$

(ii) $f * g = g * f$

(iii) $\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$

Proof. (i) Observe $\sup_x |x|^l |g(x-y)| \leq A_l(1+|y|^l)$ (this can be checked by considering the two cases $|x| \leq 2|y|$ and $|x| \geq 2|y|$), so

$$\sup_x |x^l (f \star g)(x)| \leq A_l \int_{-\infty}^{\infty} |f(y)|(1+|y|)^l dy \leq C_l.$$

These estimates also carry over to derivatives of $(f * g)$, because

$$\begin{aligned}\frac{d^k}{dx^k}(f*g)(x) &= \frac{d^k}{dx^k}\left(\int_{-\infty}^{\infty} f(y)g(x-y)dy\right) \\ &= \int_{-\infty}^{\infty} f(y)g^{(k)}(x-y)dy \\ &= (f * g^{(k)})(x).\end{aligned}$$

(ii) We can apply a change of variables to conclude

$$(f*g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-z)g(z)dz = (g*f)(x).$$

(iii) We conpute

$$\begin{aligned}\widehat{(f*g)}(\xi) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(x-y)dye^{-2\pi i x\xi}dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)g(x-y)e^{-2\pi i(x-y)\xi}e^{-2\pi i y\xi}dxdy \\ &= \int_{-\infty}^{\infty} f(y)\int_{-\infty}^{\infty} g(x-y)e^{-2\pi i(x-y)\xi}dxe^{-2\pi i y\xi}dy \\ &= \int_{-\infty}^{\infty} f(y)\widehat{g}(\xi)e^{-2\pi i y\xi}dy \\ &= \widehat{f}(\xi)\widehat{g}(\xi),\end{aligned}$$

which completes the proof. □

Theorem 3.3.6 (Plancherel's theorem)**.** *If $f \in \mathcal{S}(\mathbf{R})$, then*

$$\|\hat{f}\|_{L^2(\mathbf{R})} = \left(\int |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} = \left(\int |f(x)|^2 dx\right)^{\frac{1}{2}} = \|f\|_{L^2(\mathbf{R})}.$$

Proof. For $f \in \mathcal{S}(\mathbf{R})$ define $f^\flat(x) = \overline{f(-x)}$. Then

$$\begin{aligned}\widehat{f^\flat}(\xi) &= \int_{-\infty}^{\infty} \overline{f(-x)}e^{-2\pi i x\xi}dx \\ &= \overline{\int_{-\infty}^{\infty} f(-x)e^{2\pi i x\xi}dx} \\ &= \overline{\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx} \\ &= \overline{\hat{f}(\xi)}.\end{aligned}$$

Now let $h = f * f^\flat$. Clearly then

$$\widehat{h}(\xi) = \widehat{(f*f^\flat)}(\xi) = \widehat{f}(\xi)\widehat{f^\flat}(\xi) = |\widehat{f}(\xi)|^2$$

and

$$h(0) = \int f(y)f^\flat(-y)dy = \int f(y)\overline{f(y)}dy = \int |\widehat{f}(\xi)|^2 dy.$$

So, by Theorem 3.3.2,

$$\int |f(y)|^2 dy = h(0) = \int \hat{h}(\xi)dy = \int |\hat{f}(\xi)|^2 dy,$$

as claimed. □

Remark 3.3.7. *We could easily generalise Theorem 3.3.6 to $f \in \mathcal{M}(\mathbf{R})$. In fact, as you might imagine, we can go further and use Theorem 3.3.6 to make sense of the Fourier transform for functions in $L^2(\mathbf{R})$, but will not do this here.*

Chapter 4

Applications

4.1 The Schrödinger Equation

An important equation in the study of quantum mechanics is the Schrödinger equation. A function $\psi\colon \mathbf{R}\times[0,\infty)\to\mathbf{C}$ solves the Schrödinger equation when

$$i\partial_t\psi(x,t)-\Delta\psi(x,t)=0, \tag{4.1}$$

where ∂_t is the first-order partial derivative in the second variable and $\Delta=\partial_x^2$ is the second-order partial derivative in the first variable. The differential operator Δ is called the Laplacian.

Given data $\psi_0\in\mathcal{S}(\mathbf{R})$ we wish to find a solution ψ to (4.1) such that

$$\psi(x,0)=\psi_0(x) \tag{4.2}$$

for all $x\in\mathbf{R}$. In order to do this, we take the Fourier transform of equation (4.1) in the first variable to obtain

$$i\partial_t\widehat{\psi}(\xi,t)+4\pi^2|\xi|^2\widehat{\psi}(\xi,t)=0, \tag{4.3}$$

where $\widehat{\psi}$ denotes the Fourier transform in the first variable. Taking the Fourier transform of the initial condition (4.2) we see we require $\widehat{\psi}(\xi,0)=\widehat{\psi_0}(\xi)$. For each ξ (4.3) is an ordinary differential equation in t, so we can find at least one solution $\widehat{\psi}$ which satifies both of these conditions by requiring

$$\widehat{\psi}(\xi,t)=e^{i4\pi^2|\xi|^2t}\widehat{\psi_0}(\xi).$$

Applying Theorem 3.3.2 to this gives us that

$$\psi(x,t)=\int_{-\infty}^{\infty}e^{i4\pi^2|\xi|^2t}\widehat{\psi_0}(\xi)e^{2\pi ix\xi}d\xi. \tag{4.4}$$

This formula allows us to prove the following existence result.

Theorem 4.1.1. *Fix $\psi_0\in\mathcal{S}(\mathbf{R})$. The function $\psi\colon\mathbf{R}\times[0,\infty)\to\mathbf{C}$ defined by (4.4) is infinitely differentiable in both variables and is such that*

$$\partial_t^k\psi\in\mathcal{S}(\mathbf{R})$$

for each $k\in\mathbf{N}$. Moreover, ψ safisfies (4.1) and (4.2).

Proof. That ψ is infinitely differentiable will be proved as a homework exercise. Let us assume that we can interchange differentiation with respect to x and t with integration in ξ in formula (4.4). This will also be a homework exercise. We have that

$$\begin{aligned}\left|x^m\partial_x^l(\partial_t^k\psi)(x,t)\right|&=\left|\int_{-\infty}^{\infty}(4\pi^2i|\xi|^2)^k(2\pi i\xi)^l\widehat{\psi}_0(\xi)\frac{\partial_\xi^m(e^{2\pi ix\xi})}{(2\pi i)^m}d\xi\right|\\&=\left|\frac{1}{(-2\pi i)^m}\int_{-\infty}^{\infty}\partial_\xi^m\left[(4\pi^2i|\xi|^2)^k(2\pi i\xi)^l\widehat{\psi}_0(\xi)\right]e^{2\pi ix\xi}d\xi\right|\le C_{k,l,m},\end{aligned}$$

which proves that $\partial_t^k\psi\in\mathcal{S}(\mathbf{R})$.

By Theorem 3.3.2,

$$\psi(x,0)=\int_{-\infty}^{\infty}\widehat{\psi_0}(\xi)e^{2\pi ix\xi}d\xi=\psi_0(x)$$

and

$$i\partial_t\psi(x,t)-\Delta\psi(x,t)=\int_{-\infty}^{\infty}(-4\pi^2|\xi|^2+4\pi^2|\xi|^2)e^{i4\pi^2|\xi|^2t}\widehat{\psi_0}(\xi)e^{2\pi ix\xi}d\xi=0,$$

so ψ satisfies (4.2) and (4.1). □

For each $t\in[0,\infty)$, ψ defined by (4.4) belongs to $\mathcal{S}(\mathbf{R})$ and ψ is infinitely differentiable in both variables. (As an exercise, check this for yourself.)

To help us prove uniqueness of solutions we need the following result, which is interesting in its own right.

Lemma 4.1.2. *If ψ satisfies (4.1) then*

$$\int_{-\infty}^{\infty}|\psi(x,t)|^2dx=\int_{-\infty}^{\infty}|\psi(x,0)|^2dx,$$

for all $t\ge0$.

Proof. We define

$$E(t):=\int_{-\infty}^{\infty}|\psi(x,t)|^2dx,$$

for $t\ge0$. It will suffice to show that E is constant. To do this we compute

$$\begin{aligned}\frac{dE}{dt}(t)&=\int_{-\infty}^{\infty}(\psi(x,t)\overline{\partial_t\psi(x,t)}+\overline{\psi(x,t)}\partial_t\psi(x,t))dx\\&=-\int_{-\infty}^{\infty}(\psi(x,t)\overline{i\Delta\psi(x,t)}+\overline{\psi(x,t)}i\Delta\psi(x,t)dx\\&=-i\int_{-\infty}^{\infty}(\nabla\psi(x,t)\overline{\nabla\psi(x,t)}-\overline{\nabla\psi(x,t)}\nabla\psi(x,t)dx=0,\end{aligned}$$

which proves that E is indeed constant. □

Theorem 4.1.3 (Uniqueness of solutions)**.** *Suppose that $\tilde{\psi}\colon\mathbf{R}\times[0,\infty)\to\mathbf{C}$ is infinitely differentiable and $\partial_t^k\tilde{\psi}\in\mathcal{S}(\mathbf{R})$ for each $t\ge0$. If $\tilde{\psi}$ satisfies (4.2) and (4.1), then*

$$\tilde{\psi}=\psi$$

where ψ is defined by (4.4).

Proof. Observe that $\tilde{\psi}-\psi$ solves (4.1) with $\tilde{\psi}(x,0)-\psi(x,0)=0$ for all $x\in\mathbf{R}$. Hence, by Lemma 4.1.2,

$$\int_{-\infty}^{\infty}|\tilde{\psi}(x,t)-\psi(x,t)|^2dx=\int_{-\infty}^{\infty}|\tilde{\psi}(x,0)-\psi(x,0)|^2dx=0.$$

Therefore, as both $\tilde{\psi}$ and ψ are continuous, $\tilde{\psi}-\psi\equiv0$. □

4.2 The Heisenberg Uncertainty Principle

Theorem 4.2.1. *Suppose ψ is a function in $\mathcal{S}(\mathbf{R})$ which satisfies the normalisation condition $\|\psi\|_{L^2(\mathbf{R})}=1$. Then*

$$\left(\int_{-\infty}^{\infty}x^2|\psi(x)|^2dx\right)\left(\int_{-\infty}^{\infty}\xi^2|\widehat{\psi}(\xi)|^2d\xi\right)\ge\frac{1}{16\pi^2}$$

and equality holds if and only if $\psi(x)=Ae^{-Bx^2}$ with $B>0$ and $|A|^2=\sqrt{2B/\pi}$.

In fact,

$$\left(\int_{-\infty}^{\infty}(x-x_0)^2|\psi(x)|^2dx\right)\left(\int_{-\infty}^{\infty}(\xi-\xi_0)^2|\widehat{\psi}(\xi)|^2d\xi\right)\geq\frac{1}{16\pi^2}$$

for every $x_0,\xi_0\in\mathbf{R}$.

Proof. Integrating by parts, we have

$$\begin{aligned}1&=\int_{-\infty}^{\infty}|\psi(x)|^2dx\\&=-\int_{-\infty}^{\infty}x\frac{d}{dx}\left(|\psi(x)|^2\right)dx\\&=-\int_{-\infty}^{\infty}x\left(\psi'(x)\overline{\psi(x)}+\psi(x)\overline{\psi'(x)}\right)dx\\&\leq 2\int_{-\infty}^{\infty}|x||\psi'(x)||\psi(x)|dx\\&\leq 2\left(\int_{-\infty}^{\infty}x^2|\psi(x)|^2dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}|\psi'(x)|^2dx\right)^{\frac{1}{2}}.\end{aligned}$$

But, by Proposition 3.1.6,

$$\int_{-\infty}^{\infty}|\psi'(x)|^2dx=\int_{-\infty}^{\infty}4\pi^2\xi^2|\widehat{\psi}(\xi)|^2d\xi.$$

Combining these two calculations proves the first statement.

If we have equality, then equality must hold where we applied the Cauchy-Schwarz inequality. This happens exactly when ψ' and $\cdot\psi(\cdot)$ are linearly dependent, that is $\psi'(x)=\beta x\psi(x)$ for some $\beta\in\mathbf{C}$. The general form of solutions to this equation is

$$\psi(x)=Ae^{\beta x^2/2}.$$

Since we require $\psi\in\mathcal{S}(\mathbf{R})$, we must take $\beta=2B>0$ and the normalisation condition $\|\psi\|_{L^2(\mathbf{R})}=1$ means that $|A|^2=\sqrt{2B/\pi}$. □

4.3 Laplace's Equation

We now study *Laplace's equation*

$$\frac{\partial^2u}{\partial x^2}(x,y)+\frac{\partial^2u}{\partial y^2}(x,y)=0, \tag{4.5}$$

for $x\in\mathbf{R}$ and $y>0$. We denote the upper half-plane as $\mathbf{R}^2_+=\{(x,y)\,|\,x\in\mathbf{R},y>0\}$. We will often write

$$\Delta u:=\frac{\partial^2u}{\partial x^2}(x,y)+\frac{\partial^2u}{\partial y^2}(x,y),$$

where the operator Δ is called the *Laplacian*, or *Laplace's operator*. This is not to be confused with our earlier use of this notation. For a function u of n-variables,

$$\Delta u:=\sum_{i=1}^{n}\frac{\partial^2u}{\partial x_i^2}(x_1,x_2,\ldots,x_n),$$

so earlier we employed the notation with a function of one spatial variable, and here we employ it with a function of two.

Once again we wish to study solutions to equation (4.5) in $\mathbf{R}^2_+$ subject to the boundary condition

$$u(x,0)=f(x) \tag{4.6}$$

for all $x\in\mathbf{R}$ and a given $f\in\mathcal{S}(\mathbf{R})$. We take the Fourier transform of (4.5) in the x-variable and obtain

$$4\pi^2\xi^2\widehat{u}(\xi,y)+\frac{\partial^2\widehat{u}}{\partial y^2}(x,y)=0,$$

where $\widehat{u}$ denotes the Fourier transform of u in the x-variable. The boundary condition (4.6) becomes

$$\widehat{u}(\xi,0)=\widehat{f}(\xi),$$

for all $\xi\in\mathbf{R}$. The general solution of this ordinary differential equation in the y-variable (with ξ fixed) is

$$\widehat{u}(\xi,y)=A(\xi)e^{-2\pi|\xi|y}+B(\xi)e^{2\pi|\xi|y}.$$

We disregard the second term as it is exponentially increasing. Taking into account (4.6) we see that

$$\widehat{u}(\xi,y)=\widehat{f}(\xi)e^{-2\pi|\xi|y}.$$

Lemma 4.3.1. *The following identities hold:*

$$\int_{-\infty}^{\infty}e^{-2\pi|\xi|y}e^{2\pi i\xi x}d\xi=P_y(x)$$

and

$$\int_{-\infty}^{\infty}P_y(x)e^{2\pi i\xi x}dx=e^{-2\pi|\xi|y},$$

where

$$P_y(x)=\frac{y}{\pi(x^2+y^2)}$$

is called the Poisson kernel.

Proof. We have

$$\begin{aligned}\int_0^{\infty}e^{-2\pi|\xi|y}e^{2\pi i\xi x}d\xi&=\int_0^{\infty}e^{2\pi i(x+iy)\xi}d\xi\\&=\left.\frac{e^{2\pi i(x+iy)\xi}}{2\pi i(x+iy)}\right|_0^{\infty}=\frac{-1}{2\pi i(x+iy)}\end{aligned}$$

and

$$\int_{-\infty}^{0}e^{-2\pi|\xi|y}e^{2\pi i\xi x}d\xi=\frac{1}{2\pi i(x-iy)},$$

so

$$\int_{-\infty}^{\infty}e^{-2\pi|\xi|y}e^{2\pi i\xi x}d\xi=\frac{-1}{2\pi i(x+iy)}+\frac{1}{2\pi i(x-iy)}=\frac{y}{\pi(x^2+y^2)}.$$

This proves the first identity. The second identity is a consequence of the Fourier Inversion formula once we observe the proof of Theorem 3.3.2 generalises to functions of moderate decrease. □

Lemma 4.3.2. *The Poisson kernel* $\{P_y\}_{y>0}$ *is a good kernel as* $y\to0$.

Proof. Setting $\xi=0$ in the second identity of Lemma 4.3.1 shows that

$$\int_{\mathbf{R}}P_y(x)dx=1.$$

Clearly $P_y\geq0$ so it only remains to check the last property.

$$\begin{aligned}\int_{\delta}^{\infty}\frac{y}{\pi(x^2+y^2)}dx&=\int_{\delta/y}^{\infty}\frac{1}{\pi(1+u^2)}du\\&=\arctan(u)\Big|_{\delta/y}^{\infty}=\frac{\pi}{2}-\arctan(\delta/y)\to0\end{aligned}$$

as $y\to0$. As P_y is an even function, this completes the proof. □

Theorem 4.3.3. *Given $f \in \mathcal{S}(\mathbf{R})$, let $u(x,y) = (f * P_y)(x)$. Then*

1. *u is twice-continuously differentiable in $\mathbf{R}^2_+$ and $\Delta u = 0$;*
2. *$u(x,y) \to f(x)$ uniformly as $y \to 0$;*
3. *$\int_{\mathbf{R}} |u(x,y) - f(x)|^2 dx \to 0$ as $y \to 0$;*
4. *if $u(x,0) = f(x)$ for all $x \in \mathbf{R}$, then u is continuous in $\overline{\mathbf{R}^2_+}$ and $u(x,y) \to 0$ as $|x| + y \to \infty$.*

Proof. Differentiating under the integral sign, we can check that $u \in C^2(\mathbf{R}^2_+)$ and $\Delta u = 0$, which proves 1. Part 2 follows immediately from Lemma 4.3.2. To prove 3, we compute, for a fixed $\varepsilon > 0$,

$$\begin{aligned}\int_{\mathbf{R}} |u(x,t) - f(x)|^2 dx &= \int_{\mathbf{R}} |\widehat{u}(\xi, y) - \widehat{f}(\xi)|^2 d\xi \\ &= \int_{\mathbf{R}} |\widehat{f}||e^{-4\pi y|\xi|} - 1|^2 d\xi \\ &= \int_{|\xi|<N} |\widehat{f}||e^{-4\pi y|\xi|} - 1|^2 d\xi + \int_{|\xi|>N} |\widehat{f}||e^{-4\pi y|\xi|} - 1|^2 d\xi \\ &\le \int_{|\xi|<N} C|e^{-4\pi y|\xi|} - 1|^2 d\xi + \int_{|\xi|>N} 4|\widehat{f}| d\xi \\ &< \varepsilon,\end{aligned}$$

for N sufficiently large and y sufficiently close to 0, where we used the fact that $\widehat{f}$ is bounded and $|e^{-4\pi y|\xi|} - 1| \le 2$.

Finally, to prove 4, we have $\sup_x |P_y(x)| \le C/y$ and so $|(f * P_y)(x)| \le C/y$. Also

$$\begin{aligned}|(f * P_y)(x)| &= \left|\int_{|t|<|x|/2} f(xt) P_y(t) dt\right| + \left|\int_{|t|>|x|/2} f(x-t) P_y(t) dt\right| \\ &\le C\left(\frac{|x|}{2x^2} + \frac{y}{x^2+y^2}\right) = C\left(\frac{1}{2|x|} + \frac{y}{x^2+y^2}\right).\end{aligned}$$

Now, applying the first estimate when $|x| \le y$ and the second when $|x| \ge y$ completes the proof. □

4.4 Connections with analytic functions

We have seen that given a function $f \in \mathcal{S}(\mathbf{R})$ we can find a function $u\colon \mathbf{R}^2_+ \to \mathbf{R}$ such that $\Delta u = 0$ and $u(x,0) = f(x)$ for all $x \in \mathbf{R}$. This function was written $u(x,y) = (P_y * f)(x)$. It is natural to ask how might one represent the harmonic conjugate to u. That is, how might one represent a real-valued function v such that $F = u + iv$ is analytic in $\mathbf{R}^2_+ = \{z \in \mathbf{C} \,|\, \mathrm{Im}(z) > 0\}$? And in particular, what can one say about the boundary value of v?

Observe

$$\begin{aligned}(P_y * f)(x) &= \frac{y}{\pi}\int_{\mathbf{R}} \frac{f(t)}{(x-t)^2 + y^2} dt \\ &= \mathrm{Re}\left(\frac{i}{\pi}\int_{\mathbf{R}} \frac{f(t)}{(x-t)+iy} dt\right) \\ &= \mathrm{Re}\left(\frac{i}{\pi}\int_{\mathbf{R}} \frac{f(t)}{z-t} dt\right), \quad \text{with } z = x + iy.\end{aligned}$$

Now $F_f\colon \mathbf{R}^2_+ \to \mathbf{C}$ defined by

$$F_f(z) = \frac{i}{\pi}\int_{\mathbf{R}} \frac{f(t)}{z-t} dt$$

is analytic, since $F_f'(z) = 0$ (check this). Thus a harmonic conjugate of u is

$$v(x,y) = \mathrm{Im}\left(\frac{i}{\pi}\int_{\mathbf{R}} \frac{f(t)}{z-t} dt\right) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{f(t)(x-t)}{(x-t)^2 - y^2} dt = (f * Q_y)(x),$$

where

$$Q_y(x) = \frac{x}{\pi(x^2+y^2)}$$

is the *conjugate Poisson kernel* and again $z = x + iy$.

We now introduce a family of functions $H_h(f)$, parametrised by $h > 0$, that we will show below is closely related to the boundary value of v. For $h > 0$ we define the *truncated Hilbert transform* $H_h(f)$ of $f \in \mathcal{S}(\mathbf{R})$ by

$$H_h(f)(x) = \frac{1}{\pi}\int_{h<|x-t|<1/h} \frac{f(t)}{x-t} dt.$$

Theorem 4.4.1. *For any $f \in \mathcal{S}(\mathbf{R})$ we have that*

$$(f * Q_h)(x) - H_h(f)(x) \to 0$$

for all $x \in \mathbf{R}$ as $h \to 0$.

Proof. First

$$\begin{aligned}&(f * Q_h)(x) - H_h(f)(x) \\ &= (f * Q_h)(x) - \frac{1}{\pi}\int_{h<|x-t|<1/h} \frac{f(t)}{x-t} dt \\ &= (f * Q_h)(x) - \frac{1}{\pi}\int_{h<|x-t|} \frac{f(t)}{x-t} dt + \frac{1}{\pi}\int_{1/h<|x-t|} \frac{f(t)}{x-t} dt \\ &= (f * \psi_h)(x) + \frac{1}{\pi}\int_{1/h<|x-t|} \frac{f(t)}{x-t} dt,\end{aligned}$$

where

$$\psi(t) = \begin{cases} \frac{t}{\pi(t^2+1)} - \frac{1}{\pi t}, & \text{when } |t| \ge 1; \\ \frac{t}{\pi(t^2+1)}, & \text{when } |t| < 1. \end{cases}$$

Clearly $\int_{1/h<|x-t|} \frac{f(t)}{x-t} dt \to 0$ as $h \to 0$. Also we can easily check that $\{\psi_h\}_{h>0}$ satisfies the properties of being a good kernel, except that the first condition is replaced with the equality

$$\int_{\mathbf{R}} \psi_h(t) dt = 0$$

for all $h > 0$. Repeating the proof of Theorem 3.2.5 we see that $(f * \psi_h)(x) \to 0$ as $h \to 0$ and so complete the proof. □

Now we consider the limit

$$H(f) := \lim_{h\to 0} H_h(f). \tag{4.7}$$

We will show this limit exists as a limit in $L^2(\mathbf{R})$ for each $f \in \mathcal{S}(\mathbf{R})$ and moreover

$$\|H(f)\|_{L^2(\mathbf{R})} \le C\|f\|_{L^2(\mathbf{R})} \quad \text{for all } f \in \mathcal{S}(\mathbf{R}), \tag{4.8}$$

where $C > 0$ does not depend on f. This inequality means that H defined above for $f \in \mathcal{S}(\mathbf{R})$ may be extended to an operator from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$. We call H the *Hilbert transform.*

Theorem 4.4.2. *For each $f \in \mathcal{S}(\mathbf{R})$, the limit in (4.7) exists in $L^2(\mathbf{R})$ and there exists a constant $C > 0$ such that (4.8) holds.*

Proof. First of all, observe that $H_h(f)$ is the convolution of f with

$$x \mapsto \begin{cases} \frac{1}{\pi x}, & \text{for } h < |x| < 1/h; \\ 0, & \text{otherwise.} \end{cases}$$

Let us compute

$$\begin{aligned} &\int_{h<|x|<M} \frac{1}{\pi x} e^{2\pi i x\xi} dx = 2i \int_h^M \frac{\sin(2\pi x\xi)}{\pi x} dx \\ &= 2i \int_0^M \frac{\sin(2\pi x\xi)}{\pi x} dx - 2i \int_0^h \frac{\sin(2\pi x\xi)}{\pi x} dx \\ &= \frac{2i}{\pi} \int_0^{2\pi M\xi} \frac{\sin(y)}{y} dy - \frac{2i}{\pi} \int_0^{2\pi h\xi} \frac{\sin(y)}{y} dy \\ &= i\left(\mathrm{Si}(2\pi M\xi) - \mathrm{Si}(2\pi h\xi)\right), \end{aligned}$$

where

$$\mathrm{Si}(w) = \frac{2}{\pi} \int_0^w \frac{\sin(y)}{y} dy$$

for all $w \in \mathbf{R}$. Our homework exercises have shown that $\lim_{w\to\pm\infty} \mathrm{Si}(w) = \pm 1$ and clearly Si is a continuous function, therefore it is also bounded. Thus, in particular, m_h is bounded where

$$m_h(\xi) = \int_{h<|x|<1/h} \frac{1}{\pi x} e^{2\pi i x\xi} dx = i\left(\mathrm{Si}(2\pi\xi/h) - \mathrm{Si}(2\pi\xi h)\right) \quad \text{for } \xi \in \mathbf{R}. \tag{4.9}$$

We leave it as an exercise to check that, for $f \in \mathcal{S}(\mathbf{R})$ and k piecewise continuous and zero outside a bounded interval,

$$(f * k)\widehat{\ }(\xi) = \widehat{k}(\xi)\widehat{f}(\xi).$$

This means that $\widehat{H_h(f)}(\xi) = m_h(\xi)\widehat{f}(\xi)$ for all $\xi \in \mathbf{R}$. Putting these facts together, we find that

$$\|H_h(f)\|_{L^2(\mathbf{R})} = \|\widehat{H_h(f)}\|_{L^2(\mathbf{R})} = \|m_h\widehat{f}\|_{L^2(\mathbf{R})} \le C\|\widehat{f}\|_{L^2(\mathbf{R})} = C\|f\|_{L^2(\mathbf{R})}, \tag{4.10}$$

where C is a bound on m_h.

Now we will show that $H_h(f)$ converges in $L^2(\mathbf{R})$ as $h \to 0$. Indeed, for a fixed $\varepsilon > 0$, δ sufficiently small and M sufficiently large,

$$\begin{aligned} &\|H_h(f) - H_l(f)\|^2_{L^2(\mathbf{R})} = \|m_h\widehat{f} - m_l\widehat{f}\|^2_{L^2(\mathbf{R})} \\ &= \int_{|\xi|<\delta} |m_h(\xi) - m_l(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi + \int_{\delta\le|\xi|<M} |m_h(\xi) - m_l(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi \\ &\quad + \int_{M\le|\xi|} |m_h(\xi) - m_l(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi \\ &= \frac{\varepsilon}{3} + \int_{\delta\le|\xi|<M} |m_h(\xi) - m_l(\xi)|^2 |\widehat{f}(\xi)|^2 d\xi + \frac{\varepsilon}{3} \\ &= \frac{2\varepsilon}{3} + \sup_{\delta\le|\xi|<M} |\mathrm{Si}(2\pi\xi/h) - \mathrm{Si}(2\pi\xi/l)|^2 \|\widehat{f}\|^2_{L^2(\mathbf{R})} + \sup_{\delta\le|\xi|<M} |\mathrm{Si}(2\pi\xi h) - \mathrm{Si}(2\pi\xi l)|^2 \|\widehat{f}\|^2_{L^2(\mathbf{R})}, \end{aligned}$$

using (4.9). This can be made less than ε for h and l sufficiently close to zero. Thus $\{H_h(f)\}_{h>0}$ is a Cauchy sequence as $h \to 0$ and so (4.7) exists in $L^2(\mathbf{R})$.

Finally, to prove (4.8),

$$\|H(f)\|_{L^2(\mathbf{R})} \le \|H(f) - H_h(f)\|_{L^2(\mathbf{R})} + \|H_h(f)\|_{L^2(\mathbf{R})} \le C\|f\|_{L^2(\mathbf{R})} + \|H(f) - H_h(f)\|_{L^2(\mathbf{R})},$$

by (4.10). However, as we have just proved $H_h(f)$ converges to $H(f)$ in $L^2(\mathbf{R})$ as $h \to 0$, the second norm on the right-hand side tends to zero as $h \to 0$, so if fact we have (4.8). □