Fourier Analysis
Second Semester 2009/10
Homework Assignment 3
(Due on 23rd February 2010)

Only the questions marked with an asterisk (*) will count towards the assessment for this course.

Fourier Series.

1. Prove that $l^2(\mathbf{Z})$ is complete, that is, every Cauchy sequence in $l^2(\mathbf{Z})$ has a limit in $l^2(\mathbf{Z})$.

*2. Consider the function $f\colon [-\pi,\pi] \to \mathbf{C}$ defined by $f(\theta) = |\theta|$. Use Parseval's Identity to prove

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

3. In addition to what we discovered in question 2, it is, in fact, possible to calculate a formula for $\sum_n 1/n^k$ for every even k. However, nobody has yet managed to do this for odd k, even for $k=3$. If you have some spare time, perhaps you might want to try to evaluate $\sum_n 1/n^3$. If you succeed you can publish your first paper.

4. Suppose that f is a continuously differentiable 2π-periodic function such that

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0.$$

(a) Use Parseval's Identity to show that

$$\int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta \le \int_{-\pi}^{\pi} |f'(\theta)|^2\, d\theta.$$

(b) Find a function f_0 which satisfies the same conditions as f but

$$\int_{-\pi}^{\pi} |f_0(\theta)|^2\, d\theta = \int_{-\pi}^{\pi} |f_0'(\theta)|^2\, d\theta.$$

(c) Find a function f_1 which satisfies the same conditions as f but

$$\int_{-\pi}^{\pi} |f_1(\theta)|^2\, d\theta < \int_{-\pi}^{\pi} |f_1'(\theta)|^2\, d\theta.$$

(d) Suppose g is continuously differentiable and f is as above. Prove that

$$\left|\int_{-\pi}^{\pi} \overline{f(\theta)} g(\theta)\, d\theta\right|^2 \le \int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta \int_{-\pi}^{\pi} |g'(\theta)|^2\, d\theta.$$

(e) Suppose h is continuously differentiable on $[0,\pi]$ and $h(0) = h(\pi) = 0$. Prove

$$\int_{0}^{\pi} |h(\theta)|^2\, d\theta \le \int_{0}^{\pi} |h'(\theta)|^2\, d\theta.$$

5. Suppose f is 2π-periodic function which is k-times continuously differentiable. Prove that

$$\hat{f}(n) = O(1/|n|^k)$$

as $|n| \to \infty$.

6. Suppose f is 2π-periodic and integrable on $[-\pi,\pi]$.

(a) Show that

$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x+\pi/n)]e^{-inx}\, dx.$$

(b) Assume that f is Hölder continuous of order α, that is

$$|f(x) - f(y)| \le C|x-y|^{\alpha}$$

for some $C > 0$ and all x and y. Use part (a) to prove

$$\hat{f}(n) = O(1/|n|^{\alpha}).$$

The Fourier transform.

7. Prove the following proposition, which we stated in class.

   Proposition. *If* $f \in \mathcal{S}(\mathbf{R})$ *then*
   (i) $f(x+h) \to \widehat{f}(\xi)e^{2\pi i h\xi}$ *whenever* $h \in \mathbf{R}$,
   (ii) $f(x)e^{-2\pi i x h} \to \widehat{f}(\xi + h)$ *whenever* $h \in \mathbf{R}$,
   (iii) $f(\delta x) \to \delta^{-1}\widehat{f}(\delta^{-1}\xi)$ *for* $\delta > 0$,
   (iv) $f'(x) \to 2\pi i\xi\widehat{f}(\xi)$,
   (v) $-2\pi i x f(x) \to (\widehat{f})'(\xi)$.

8. The aim of this question is to prove that $g\colon \mathbf{R} \to \mathbf{C}$ defined by
$$g(x) = e^{-\pi x^2}$$
for all $x \in \mathbf{R}$ is equal to its own Fourier transform.
   (a) By a simple calculation, check that $g'(x) = -2\pi x g(x)$.
   (b) Set
$$G(\xi) = \widehat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i x\xi}\, dx$$
   and use the above fact to prove that $G'(\xi) = -2\pi\xi G(\xi)$.
   (c) Define $H(\xi) = G(\xi)e^{\pi\xi^2}$ and show that $H'(\xi) = 0$ for all $\xi \in \mathbf{R}$.
   (d) Conclude that $G(\xi) = e^{-\pi\xi^2}$. You may assume $G(0) = 1$ without proof.

9. Let f and g be functions defined by
$$f(x) = \begin{cases} 1, & \text{if } |x| \le 1, \\ 0, & \text{if } |x| > 1, \end{cases}$$
and
$$g(x) = \begin{cases} 1 - |x|, & \text{if } |x| \le 1, \\ 0, & \text{if } |x| > 1. \end{cases}$$
Although f is not continuous, observe the integral defining the Fourier transform of f still makes sense. Show that
$$\widehat{f}(\xi) = \frac{\sin(2\pi\xi)}{\pi\xi} \quad \text{and} \quad \widehat{g}(x) = \left(\frac{\sin(2\pi\xi)}{\pi\xi}\right)^2,$$
for $\xi \neq 0$ and compute $\widehat{f}(0)$ and $\widehat{g}(0)$.

*10. Let f be a function of moderate decrease. This question illustrates the principle that decay in the Fourier transform $\widehat{f}$ is related to the continuity of the function f. Suppose that $\widehat{f}$ is continuous and satisfies
$$|\widehat{f}(\xi)| \le \frac{C}{|\xi|^{1+\alpha}}$$
for some $C > 0$ and $0 < \alpha < 1$.
   (a) Derive an expression for $f(x+h) - f(x)$ using the inversion formula for the Fourier transform. This expression should be an integral of some kind.
   (b) Split the integral in two, integrating over $|\xi| \le 1/|h|$ and $|\xi| > 1/|h|$. Estimate each integral separately to conclude
$$|f(x+h) - f(x)| \le A|h|^{\alpha}$$
   for some $A > 0$.

11. Suppose f is a function of moderate decrease (hence, also continuous).
   (a) Prove that $\widehat{f}$ is continuous and that $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$ by proving the formula
$$\widehat{f}(\xi) = \frac{1}{2}\int [f(x) - f(x - 1/(2\xi)]e^{-2\pi i x\xi}\, d\xi.$$
   (b) Show that if $\widehat{f}(\xi) = 0$ for all $\xi \in \mathbf{R}$, then $f(x) = 0$ for all $x \in \mathbf{R}$. [Hint: Check the multiplication formula $\int \widehat{f}(y)g(y)\, dy = \int f(y)\widehat{g}(y)\, dy$ is valid for such f when $g \in \mathcal{S}(\mathbf{R})$.]