Fourier Analysis
Second Semester 2009/10
Homework Assignment 2
(Due on 9th February 2010, please staple muliple sheets together)

Only the questions marked with an asterisk (*) will count towards the assessment for this course. Most of these exercises are taken from Stein and Shakarchi.

*1. For $\delta \in (0, \pi)$, let f be defined on $[-\pi, \pi]$ by

$$f(\theta) = \begin{cases} 0, & \text{if } |\theta| > \delta, \\ 1 - |\theta|/\delta, & \text{if } |\theta| \leq \delta. \end{cases}$$

(a) Plot the graph of f.
(b) Show that

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1-\cos(n\delta)}{n^2\pi\delta} \cos(n\theta).$$

2. Let f be the function given by $f(\theta) = |\theta|$ for $\theta \in [-\pi, \pi]$.
(a) Draw the graph of f.
(b) Calculate the Fourier coefficients of f and show that

$$\hat{f}(n) = \begin{cases} \pi/2, & \text{if } n = 0, \\ \{(-1)^n - 1\}/(\pi n^2), & \text{if } n \neq 0. \end{cases}$$

(c) What is the Fourier series of f in terms of sines and cosines?
(d) Taking $\theta = 0$, prove that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sum of the series $\sum_n b_n$ for $k \geq 1$ and define $B_0 = 0$.
(a) **Summation by Parts.** Prove the formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

(b) Deduce from the above formula the following lemma:
Lemma (Dirichlet's convergence test). If the partial sums of the series $\sum_n b_n$ are bounded (that is, $B_k \leq C$ for all $k \geq 1$) and $\{a_n\}_n$ is a sequence of real numbers that decrease monotonically to 0, then $\sum_n a_n b_n$ converges.

4. Consider the function $f\colon [-\pi, \pi] \to \mathbf{C}$ defined by

$$f(x) = \begin{cases} -\frac{\pi}{2} - \frac{x}{2}, & \text{if } -\pi \leq x < 0, \\ 0, & \text{if } x = 0, \\ \frac{\pi}{2} - \frac{x}{2}, & \text{if } 0 < x \leq \pi. \end{cases}$$

Draw the graph of f. Prove that the Fourier series of f is

$$\frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}$$

and prove that it converges pointwise to f, even though f is not continuous. [Hint: Use Dirichlet's convergence test.]

5. Suppose that $\{f_n\}_n$ is a sequence of integrable functions on $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} |f_k(x) - f(x)|\, dx \to 0$$

as $k \to \infty$ for another integrable function $f\colon [-\pi, \pi] \to \mathbf{C}$. Show that $\hat{f}_k(n) \to \hat{f}(n)$ uniformly in n as $k \to \infty$.

6. Let f be a continuous 2π-periodic function.
(a) Show that

$$\hat{f}(n) = \frac{-1}{2\pi} \int_{-\pi}^{\pi} f(x - (\pi/n)) e^{-inx}\, dx.$$

(b) Prove that

$$\lim_{|n|\to\infty} \hat{f}(n) = 0.$$

[Hint: A continuous periodic function is uniformly continuous. You may use this fact without proof.]

(c) Using part (b), prove that if g is an integrable function on $[-\pi, \pi]$, then

$$\lim_{|n|\to\infty} \hat{g}(n) = 0.$$

This is called the Riemann-Lebesgue Lemma.

*7. The aim of this question is to evaluate the improper Riemann integral

$$\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\, dx.$$

(a) Show that

$$\lim_{b\to\infty} \int_0^b \frac{\sin x}{x}\, dx = \lim_{N\to\infty} \int_0^\pi \frac{\sin((N+1/2)x)}{x}\, dx.$$

(b) Use the Riemann-Lebesgue Lemma (as stated in question 6) to prove

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left(\frac{1}{\sin(x/2)} - \frac{2}{x} \right) \sin((N+1/2)x)\, dx = 0.$$

(c) Use the above and the fact that

$$\int_{-\pi}^{\pi} \frac{\sin((N+1/2)x)}{\sin(x/2)}\, dx = 2\pi$$

for all $N \geq 1$ to show that $\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2$.

8. Suppose that f is an integrable 2π-periodic function. Prove that for each N

$$\int_{-\pi}^{\pi} S_N(f)(x)\, dx = \int_{-\pi}^{\pi} f(x)\, dx.$$