Fourier Analysis

Comment: *The following questions, with the exception of those parts asking to state theorems, will have been seen by the students in homework assignments, although not all of them will have counted for credit.*

1.

(a) State Jordan's Criterion for Fourier series. **[7 marks]**

(b) Let $a, b \in (-\pi, \pi)$ and $a < b$. Consider the function $f\colon [-\pi, \pi] \to \mathbf{C}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } -\pi \le x < a; \\ 1, & \text{if } a \le x < b; \\ 0, & \text{if } b \le x \le \pi. \end{cases}$$

(i) Draw the graph of f. **[2 marks]**

(ii) Calculate the Fourier coefficients of f. **[7 marks]**

(iii) Prove that

$$f(x) = \frac{b-a}{2\pi} + \sum_{n \neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi i n} e^{inx} \qquad (*)$$

for all but two values of $x \in [-\pi, \pi]$. **[5 marks]**

(iv) For which two values of x does $(*)$ fail, and what is the true value of the right-hand side of $(*)$ for those values? **[4 marks]**

Solution:

(a) Let f be an integrable function **(0 marks)** which is of bounded variation in a neighbourhood of a point x **(3 marks)**. Then

$$\lim_{N \to \infty} S_N(f)(x) = \frac{1}{2}(f(x^+) + f(x^-)),$$

(4 marks) where $S_N(f)$ is the N-th partial sum of the Fourier series of f.

(b) (i) The graph should have the correct shape **(1 mark)** and the axes should be labelled **(1 mark)**.

(ii) We defined the Fourier coefficients to be

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

(2 marks). So $\widehat{f}(0)$ is clearly $(b-a)/2\pi$ **(2 marks)**. For $n \neq 0$ we have that

$$\widehat{f}(n) = \frac{1}{2\pi} \int_a^b e^{-inx}\, dx = \frac{1}{-2\pi i n} \left[e^{-inx}\right]_a^b = \frac{e^{-ina} - e^{-inb}}{2\pi i n}$$

(3 marks).

(iii) We wish to apply Jordan's Criterion to conclude f equals it's Fourier series, as the right-hand side of $(*)$ is the Fourier series of f **(2 marks)**. Clearly f is locally of bounded variation at every x, since it is locally constant except at $x = a$ and $x = b$, but even at these points it is monotonic, hence f is locally of bounded variation at every x **(2 marks)**. Therefore,

$$\frac{f(x^+) + f(x^-)}{2} = \frac{b-a}{2\pi} + \sum_{n \neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi i n} e^{inx}$$

Finally, $f(x) = (f(x^+) + f(x^-))/2$ everywhere except for $x = a$ and $x = b$, hence by Jordon's Criterion, $(*)$ holds everywhere except $x = a$ and $x = b$ **(1 mark)**.

(iv) According to Jordan's Criterion, $(*)$ fails when $x = a$ and $x = b$ by the reasoning above **(2 marks)**. The true value is $1/2$ in both cases **(2 marks)**.

2.

(a) State Parseval's Identity and give the class of functions for which it is valid. **[7 marks]**

(b) Suppose that f is a continuously differentiable 2π-periodic function such that

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta = 0.$$

(i) Prove that $(f')\hat{}(n) = (in)\widehat{f}(n)$ for all $n \in \mathbf{Z}$. **[5 marks]**

(ii) Use Parseval's Identity to show that

$$\int_{-\pi}^{\pi} |f(\theta)|^2\, d\theta \leq \int_{-\pi}^{\pi} |f'(\theta)|^2\, d\theta.$$

[7 marks]

(iii) Find a function f_0 which satisfies the same conditions as f and is not identically zero, but

$$\int_{-\pi}^{\pi} |f_0(\theta)|^2\, d\theta = \int_{-\pi}^{\pi} |f_0'(\theta)|^2\, d\theta.$$

[3 marks]

(iv) Suppose h is continuously differentiable on $[0, \pi]$ and $h(0) = h(\pi) = 0$. Prove

$$\int_{0}^{\pi} |h(\theta)|^2\, d\theta \leq \int_{0}^{\pi} |h'(\theta)|^2\, d\theta.$$

[3 marks]

Solution:

(a) For a square integrable function g on $[-\pi, \pi]$, that is for $g \in L^2(\mathbf{T})$ **(3 marks)**, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(\theta)|^2 \, d\theta = \sum_{n\in\mathbf{Z}} |\widehat{g}(n)|^2$$

(4 marks).

(b) (i) We can use integration by parts **(2 marks)** to discover

$$\widehat{f'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\, d\theta = \frac{-in}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx = -in\widehat{f}(n)$$

(3 marks) as the boundary terms cancel.

(ii) The fact that f has zero integral means that $\widehat{f}(0)$ equals zero **(1 mark)**. Thus,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 \, d\theta = \sum_{n\neq 0} |\widehat{f}(n)|^2$$

(2 marks). Now using the fact that $1 \leq n^2$ for all $n \in \mathbf{Z}\setminus\{0\}$, we have

$$\sum_{n\neq 0} |\widehat{f}(n)|^2 \leq \sum_{n\neq 0} n^2|\widehat{f}(n)|^2$$

(2 marks). Using part (i) and Parseval's Identity again, we conclude

$$\sum_{n\neq 0} n^2|\widehat{f}(n)|^2 = \sum_{n\neq 0} |\widehat{f'}(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(\theta)|^2 \, d\theta$$

(2 marks).

(iii) The only inequality we used above was $1 \leq n^2$, so we need to replace this with equality, that is $1 = n^2$ **(1 mark)**. Thus our non-zero function f_0 must be a function whose Fourier coefficients are all zero except for $n = -1, 1$. Thus $f_0(\theta) = ae^{i\theta} + be^{-i\theta}$ will do the job for constants $a, b \in \mathbf{C}$, provided one of a and b is non-zero **(2 marks)**.

(iv) Extend h to be an odd function on $[-\pi, \pi]$ and apply the result of part (ii) **(3 marks)**.

3.

The aim of this question is to prove that $g\colon \mathbf{R} \to \mathbf{C}$ defined by

$$g(x) = e^{-\pi x^2}$$

for all $x \in \mathbf{R}$ is equal to its own Fourier transform.

(a) Prove that $\widehat{f'}(\xi) = 2\pi i\xi\widehat{f}(\xi)$ for any $f \in \mathcal{S}$. **[5 marks]**

(b) By a simple calculation, check that $g'(x) = -2\pi x g(x)$. **[5 marks]**

(c) Set

$$G(\xi) = \widehat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i x\xi}\,dx$$

and use the above facts to prove that $G'(\xi) = -2\pi\xi G(\xi)$. **[5 marks]**

(d) Define $H(\xi) = G(\xi)e^{\pi\xi^2}$ and show that $H'(\xi) = 0$ for all $\xi \in \mathbf{R}$. **[5 marks]**

(e) Conclude that $G(\xi) = e^{-\pi\xi^2}$. You may assume $G(0) = 1$ without proof. **[5 marks]**

Solution:

(a) Integrating by parts, we have

$$\int_{-N}^{N} f'(x)e^{-2\pi i x\xi}\,dx = -\int_{-N}^{N} f(x)(-2\pi i\xi)e^{-2\pi i x\xi}\,dx + \left[f(x)e^{-2\pi i x\xi}\right]_{-N}^{N}$$

(3 marks). Taking the limit as $N \to \infty$ we see that the boundary terms go to zero as f is a Schwarz function. We are left with

$$\int_{\mathbf{R}} f'(x)e^{-2\pi i x\xi}\,dx = (2\pi i\xi)\int_{\mathbf{R}} f(x)e^{-2\pi i x\xi}\,dx,$$

which is what we were asked to prove **(2 marks)**.

(b) A simple consequence of the chain rule **(5 marks)**.

(c) Passing the differentiation inside the integral **(2 marks)** and integrating by parts **(2 marks)**, we have

$$\begin{aligned} G'(\xi) &= \int_{-\infty}^{\infty} \frac{d[g(x)e^{-2\pi i x\xi}]}{d\xi}\,dx \\ &= \int_{-\infty}^{\infty} -2\pi i x g(x)e^{-2\pi i x\xi}\,dx = i\int_{-\infty}^{\infty} g'(x)e^{-2\pi i x\xi}\,dx \\ &= -i\int_{-\infty}^{\infty} g(x)(-2\pi i\xi)e^{-2\pi i x\xi}\,dx = (-2\pi\xi)G(\xi). \end{aligned}$$

An attempt should be made to justify at least one of the two procedures **(1 mark)**.

(d) By part (c), we have

$$H'(\xi) = G'(\xi)e^{\pi\xi^2} + G(\xi)(2\pi\xi)e^{\pi\xi^2} = G'(\xi)e^{\pi\xi^2} - G'(\xi)e^{\pi\xi^2} = 0$$

(5 marks).

(e) Easy **(5 marks)**.

4.

The aim of this question is to prove the *Heisenberg Uncertainty Principle*: Suppose ψ is a Schwarz function such that

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1.$$

Then

$$\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\widehat{\psi}(\xi)|^2\, d\xi\right) \geq \frac{1}{16\pi^2}. \qquad (\dagger)$$

(a) Show that, under the hypothesis on ψ,

$$1 = -\int_{-\infty}^{\infty} \left(x\psi'(x)\overline{\psi(x)} + x\overline{\psi'(x)}\psi(x)\right)\, dx. \qquad (\ddagger)$$

[Hint: Write $|\psi(x)|^2 = \frac{d(x)}{dx}|\psi(x)|^2$ and use integration by parts]

[9 marks]

(b) Use ($\ddagger$) to conclude

$$1 \leq 2\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2\, dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} |\psi'(x)|^2\, dx\right)^{\frac{1}{2}}$$

[8 marks]

(c) Use the above and what you know about the Fourier transform to complete the proof of ($\dagger$). **[8 marks]**

Solution:

(a) Using integration by parts **(3 marks)** and the fact that $|a|^2 = a\overline{a}$ **(3 marks)**, we can see that

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\psi(x)|^2\, dx = \int_{-\infty}^{\infty} \frac{d(x)}{dx}|\psi(x)|^2\, dx \\ &= -\int_{-\infty}^{\infty} x\frac{d(|\psi(x)|^2)}{dx}\, dx = -\int_{-\infty}^{\infty} \left(x\psi'(x)\overline{\psi(x)} + x\overline{\psi'(x)}\psi(x)\right)\, dx \end{aligned}$$

(3 marks).

(b) Using ($\ddagger$) and the Cauchy-Schwarz inequality **(4 marks)** we have

$$1 \leq 2\int_{-\infty}^{\infty} |x||\psi'(x)|\psi(x)|\, dx \leq 2\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2\, dx\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty} |\psi'(x)|^2\, dx\right)^{\frac{1}{2}}$$

(4 marks).

(c) By Plancherel's Theorem, we have

$$\left(\int_{-\infty}^{\infty} |\psi'(x)|^2\, dx\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} |\widehat{\psi'}(\xi)|^2\, d\xi\right)^{\frac{1}{2}}$$

(3 marks) and $\widehat{\psi'}(\xi) = 2\pi i\xi\widehat{f}(\xi)$ **(3 marks)**. Combining these two facts yields ($\dagger$) **(2 marks)**.
