Mathematical Methods for Social Scientists
Math 195 (Sec 45), Autumn 2005
Revision Sheet for Mid-term 2

This mid-term will cover everything we have covered in class so far, but clear emphasis will be placed on the material not covered in the first mid-term, that is, chapter 15. The following questions are of the style you can expect in the exam.

1. (a) Define what is means for a function $f\colon \mathbf{R}^2 \to \mathbf{R}$ of two variables to be continuous at a point (x, y)
   (b) Give an example of a function $f\colon \mathbf{R}^2 \to \mathbf{R}$ for which $x \mapsto f(x, 0)$ is continuous at 0 (as a function of one variable) and $y \mapsto f(0, y)$ is continuous at 0 (as a function of one variable) but f is not continuous (as a function of two variables) at $(0, 0)$
   (c) Prove that $\lim_{(x,y)\to(0,0)} \frac{6x^2y^2}{x^2+y^2} = 0$
2. (a) Define the two first-order partial derivatives of $f\colon \mathbf{R}^2 \to \mathbf{R}$.
   (b) State Clairaut's Theorem.
   (c) Verify that the conclusion of Clairaut's Theorem holds for the function $f(x, y) = xy^3e^x$.
   (d) The wave equation is the partial differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial^2 x}.$$

   Show that for any $f\colon \mathbf{R} \to \mathbf{R}$ both $(x, t) \mapsto f(x - ct)$ and $(x, t) \mapsto f(x + ct)$ satisfy the wave equation.
3. (a) Give the equation for the tangent plane to a surface Γ given by $z = f(x, y)$ at the point (x_0, y_0, z_0).
   (b) Find the tangent plane to $z = x^2 + 4y^2$ at the point $(2, 1, 8)$.
   (c) Find the tangent plane to $z = x \ln y$ at the point $(\pi, e, 0)$.
4. (a) State the chain rule for the function $f(x, y)$ where, in turn, $x = g(s, t)$ and $y = h(s, t)$.
   (b) Find $\partial_s f$ and $\partial_t f$ when $f(x, y) = e^y \sin x$, with $x = s^2t + t^3$ and $y = s^4t^2 + st$.
5. (a) Define the directional derivative $D_{\mathbf{u}}f$ of f in the direction $\mathbf{u}$. Define the gradient vector ∇f of f.
   (b) Give an expression relating the gradient vector and the directional derivative in a direction $\mathbf{u}$.
   (c) In which direction does the directional derivative take on its maximum value?

(d) How are the gradient vector ∇f and the normal to the tangent plane of a level set of f related?

(e) Calculate ∇f for $f(x, y) = x \sin(x - y)$.

6. (a) Define what it means for $f\colon \mathbf{R}^2 \to \mathbf{R}$ to have a local maximum at (a, b). Define what it means for $f\colon \mathbf{R}^2 \to \mathbf{R}$ to have a local minimum at (a, b).

   (b) State the second derivatives test.

   (c) Find the critical points of the function $f(x, y) = 4 + x^3 + y^3 - 3xy$ then use the second derivatives test to classify them.

7. (a) State the method of Lagrange multipliers.

   (b) Use this method to find maximum and minimum values of $f(x, y) = x^2 y$ subject to the constraint $x^2 + 2y^2 = 6$

In these questions I have tried to emphasis the less computational aspects of the material. You can expect the exam to contain lengthier and more complicated calculations. The style of the more computational parts will be similar to the questions in the textbook, so practising them would be constructive. I am aiming to make this mid-term shorter but harder than the first.