Mathematical Methods for Social Scientists
Math 195 (Sec 55), Autumn 2006
Solutions for the Revision Sheet for Mid-term 2

1. (a) A function $f\colon \mathbf{R}^2 \to \mathbf{R}$ is continuous at (x, y) if

$$\lim_{(u,v)\to(x,y)} f(u, v) = f(x, y)$$

(b) One example is

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

One can easily check that $f(x, 0) = 0$ for all $x \in \mathbf{R}$ and $f(0, y) = 0$ for all $y \in \mathbf{R}$, however

$$\lim_{(x,y)\to(0,0)} f(x, y) \neq f(0, 0) = 0,$$

since

$$\lim_{x\to 0} f(x, x) = 1/2.$$

(c) We have

$$\left|\frac{6x^2y^2}{x^2 + y^2}\right| = \frac{6x^2y^2}{x^2 + y^2} \leq \frac{6(x^2 + y^2)y^2}{x^2 + y^2} = 6y^2 \leq 6(x^2 + y^2).$$

For each $\varepsilon > 0$, set $\delta = \sqrt{\varepsilon/6}$. So we have that if $0 < \sqrt{(x - 0)^2 + (y - 0)^2} < \delta$ then

$$\left|\frac{6x^2y^2}{x^2 + y^2} - 0\right| \leq 6(x^2 + y^2) < 6\delta^2 = 6(\sqrt{\varepsilon/6})^2 = \varepsilon.$$

Therefore, by definition, $\lim_{(x,y)\to(0,0)} \frac{6x^2y^2}{x^2+y^2} = 0$.

2. (a) For $f\colon \mathbf{R}^2 \to \mathbf{R}$ the partial derivative of f with respect to x is

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

and the partial derivative of f with respect to y is

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

(b) *Clairaut's Theorem.* If $f\colon \mathbf{R}^2 \to \mathbf{R}$ is such that $\partial^2 f/\partial x\partial y$ and $\partial^2 f/\partial y\partial x$ are continuous in on a disc containing a point (x, y), then

$$\frac{\partial^2 f}{\partial x\partial y}(x, y) = \frac{\partial^2 f}{\partial y\partial x}(x, y)$$

(c) You should find that $f_{xy} = 3(1+x)y^2e^x = f_{yx}$.

(d) Setting $u(x,t) = f(x-ct)$ we can apply the chain rule to find

$$u_{xx}(x,t) = f''(x-ct)$$

and

$$u_{tt}(x,t) = c^2 f''(x-ct).$$

Thus

$$u_{tt}(x,t) = c^2 f''(x-ct) = c^2 u_{xx}(x,t)$$

and so u satisfies the wave equation. The answer is similar for $f(x+ct)$.

3. (a)

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

(b)

$$z - 8 = 4(x-2) + 8(y-1)$$

(c) Note: Point should have been (π, e, π).

$$z - \pi = (x-\pi) + \frac{\pi}{e}(y-e)$$

4. (a) If $(x,y) \mapsto f(x,y)$ is a differentiable function of x and y, and $(s,t) \mapsto g(s,t)$ and $(s,t) \mapsto h(s,t)$ are differentiable functions of s and t, then $(s,t) \mapsto f(g(s,t),h(s,t))$ is a differentiable function of s and t. We also have

$$\frac{\partial f}{\partial s}(s,t) = \frac{\partial f}{\partial x}(g(s,t),h(s,t))\frac{\partial g}{\partial s}(s,t) + \frac{\partial f}{\partial y}(g(s,t),h(s,t))\frac{\partial h}{\partial s}(s,t)$$

and

$$\frac{\partial f}{\partial t}(s,t) = \frac{\partial f}{\partial x}(g(s,t),h(s,t))\frac{\partial g}{\partial t}(s,t) + \frac{\partial f}{\partial y}(g(s,t),h(s,t))\frac{\partial h}{\partial t}(s,t).$$

(b) We have

$$\begin{aligned}\frac{\partial f}{\partial s}(s,t) &= (e^y \cos x)(2st) + (e^y \sin x)(4s^3t^2 + t)\\ &= (e^{s^4t^2+st}\cos(s^2t+t^3))(2st) + (e^{s^4t^2+st}\sin(s^2t+t^3))(4s^3t^2+t)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial s}(s,t) &= (e^y \cos x)(s^2 + 3t^2) + (e^y \sin x)(2s^4t + s)\\ &= (e^{s^4t^2+st}\cos(s^2t+t^3))(s^2+3t^2) + (e^{s^4t^2+st}\sin(s^2t+t^3))(2s^4t+s).\end{aligned}$$

5. (a) For a unit vector $\mathbf{u} = \langle a, b \rangle$, we define

$$D_{\mathbf{u}}(f)(x,y) = \lim_{h \to 0} \frac{f(x+ha, y+hb) - f(x,y)}{h}$$

and the gradient vector is

$$\nabla f = \langle \partial_x f, \partial_y f \rangle$$

(b) $D_{\mathbf{u}}(f) = \nabla f \cdot \mathbf{u}$

(c) The directional derivative attains its maximum when $\mathbf{u} = \nabla f / |\nabla f|$.

(d) The vector ∇f and the normal to the tangent plane of a level set of f are parallel.

(e) For $f(x,y) = x\sin(x-y)$, we have $\nabla f(x,y) = \langle \sin(x-y) + x\cos(x-y), -x\cos(x-y) \rangle$.

6. (a) The function $f \colon \mathbf{R}^2 \to \mathbf{R}$ has a local maximum at (a,b) if there exists a $\delta > 0$ such that $f(x,y) \le f(a,b)$ for all $(x,y) \in B_\delta(a,b) = \{(x,y) | (x-a)^2 + (y-b)^2 \le \delta^2\}$. The function $f \colon \mathbf{R}^2 \to \mathbf{R}$ has a local minimum at (a,b) if there exists a $\delta > 0$ such that $f(x,y) \ge f(a,b)$ for all $(x,y) \in B_\delta(a,b)$.

(b) Suppose all the second-order derivatives of f are continuous on a disc with centre (a,b) and suppose that $\partial_x f(a,b) = 0 = \partial_y f(a,b)$. Let

$$D(a,b) = \partial_{xx} f(a,b) \partial_{yy} f(a,b) - (\partial_{xy} f(a,b))^2.$$

Then (a) if $D(a,b) > 0$ and $\partial_{xx} f(a,b) > 0$ then $f(a,b)$ is a local minimum value of f, (b) if $D(a,b) > 0$ and $\partial_{xx} f(a,b) < 0$ then $f(a,b)$ is a local maximum value of f, and (c) if $D(a,b) < 0$, then $f(a,b)$ is neither a local minimum nor a local maximum of f.

(c) The critical points of the function $f(x,y) = 4 + x^3 + y^3 - 3xy$ satisfy

$$\begin{cases} 3x^2 - 3y &= 0 \\ 3y^2 - 3x &= 0 \end{cases}$$

so they are $(0,0)$ and $(1,1)$. We find $D(x,y) = 36xy - 9$. Thus, $D(0,0) = -9 < 0$ and so $(0,0)$ is a saddle point, and $D(1,1) = 36 - 9 > 0$ and $\partial_{xx} f(1,1) = 6 > 0$ so $(1,1)$ is a local minimum point.

7. (a) The extreme values of a function $f \colon \mathbf{R}^2 \to \mathbf{R}$ subject to the constraint $g(x,y) = k$ occur at the points (x,y) which satisfy

$$\begin{cases} \nabla f(x,y) &= \lambda \nabla g(x,y) \\ g(x,y) &= k \end{cases}$$

for some $\lambda \in \mathbf{R}$. Evaluating f at the points which satisfy this system of equations and taking the largest (smallest) produces the maximum (minimum) of f subject to the constraint $g(x,y) = k$.

(b) We must solve

$$\begin{cases} \langle 2xy, x^2 \rangle &= \langle \lambda 2x, 4y \rangle \\ x^2 + 2y^2 &= 6 \end{cases}$$

We find the solutions (x, y) are $(0, \pm\sqrt{3})$ and $(2, \pm 1)$, so

$$\max_{x^2+2y^2=6} x^2 y = 4$$

and

$$\min_{x^2+2y^2=6} x^2 y = -4.$$