Name: ______________________________

Elementary Functions and Calculus III
Math 133 (Sec 22), Spring 2004

Mid-term 1

21st April 2004

Instructions: The total time allowed for this examination is 50 minutes. The use of notes, textbooks or calculators is prohibited. Write you answer to each question in the space provided below it. Should you require more space write on the reverse of the paper, labeling your answers clearly. Write your name at the top of this sheet. The maximum number of points available for each question or part question is shown in parentheses next to the question. There are 4 questions. You should attempt as many of the questions as you can. Partial credit may be given for incomplete answers.

1. Consider the following differential equation with an initial condition.

$$\begin{cases} (x^2+9)y'(x) + 2xy(x) = x^2 + 9, & \text{for all } x \in \mathbf{R}; \\ y(0) = 0 \end{cases}$$

(10 pts) (a) Compute the integrating factor for this first-order linear differential equation.

Answer. Firstly we must divide the differential equation by the coefficient of the y' term. Since this expression is always positive, this is a legal step. This gives us

$$y'(x) + \frac{2x}{x^2+9}y(x) = 1$$

Thus to compute the integrating factor we need to find an anti-derivative to the function $x \mapsto (2x)/(x^2+9)$ which is $x \mapsto \ln(x^2+9)$. Thus the integrating factor is

$$\exp(\ln(x^2+9)) = x^2 + 9$$

(10 pts) (b) Use the above or any technique you know to find an expression for y.

Answer. Multiplying by the integrating factor and using the product rule we see that

$$\frac{d}{dx}\big((x^2+9)y(x)\big) = x^2 + 9$$

so integrating from 0 to t gives

$$\int_0^t \frac{d}{dx}\big((x^2+9)y(x)\big)\,dx = \int_0^t x^2 + 9\,dx$$

and so

$$y(t) = \frac{\frac{1}{3}t^3 + 9t}{(t^2+9)}$$

2. Use a method of your choice to find the following anti-derivatives.

(7 pts) (a) $\int \sin^5 x\, dx$

Answer. Using the identity $\sin^2 x + \cos^2 x = 1$ we have

$$\begin{aligned}\int (1-\cos^2 x)^2 \sin x\, dx &= \int (1 - 2\cos^2 x + \cos^4 x)\sin x\, dt \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.\end{aligned}$$

(6 pts) (b) $\int \frac{14e^{2x}}{5+7e^{2x}}\, dx$

Answer. Using the substitution $y = e^x$ we have

$$\int \frac{14e^{2x}}{5+7e^{2x}}\, dx = \int \frac{14y}{5+7y^2}\, dy = \ln|5+7e^{2x}| + C$$

(7 pts) (c) $\int x^{89}\sqrt{x^{90}-5}\, dx$

Answer. Using the substitution $y^2 = x^{90} - 5$

$$\int x^{89}\sqrt{x^{90}-5}\, dx = \frac{1}{90}\int 2y^2\, dy = \frac{2}{90\times 3}(x^{90}-5)^{3/2} + C.$$

(6 pts) 3. (a) State the integration by parts theorem for definite integrals.

Answer. Suppose that u and v are two continuously differentiable functions on $[a, b]$. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)|_a^b - \int_a^b u'(x)v(x)\,dx$$

(7 pts) (b) Use integration by parts to find $\int_0^{\frac{\pi}{2}} x\cos x\,dx$.

Answer. We have

$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\cos x\,dx &= x\sin x|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx \\ &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin x\,dx \\ &= \frac{\pi}{2} - 1\end{aligned}$$

(7 pts) (c) Use integration by parts twice to find $A_x(e^x \cos x)$.

Answer. Using integration by parts twice we have

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$

and so $A_x(e^x \cos x) = e^x(\cos x + \sin x)/2$.

4. Use the appropriate trigonometric identity to evaluate the following integrals.

(10 pts) (a) $\int_{-4}^{4} \frac{1}{\sqrt{16-x^2}}\,dx$

Answer. Using the substitution $x = 4\sin t$ we have

$$\int_{-4}^{4} \frac{1}{\sqrt{16-x^2}}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt = \pi$$

(10 pts) (b) $\int_{4}^{12} \frac{x}{\sqrt{x^2+9}}\,dx$ (Recall: $\sin(\cos^{-1} x) = \sqrt{1-x^2}$)

Answer. Using the substitution $x = 3\tan t$ we have

$$\int_{4}^{12} \frac{x}{\sqrt{x^2+9}}\,dx = \int_{\tan^{-1}(\frac{4}{3})}^{\tan^{-1}(\frac{12}{3})} 3\frac{\sin t}{\cos^2 t}\,dt = \left.\frac{3}{\cos t}\right|_{\tan^{-1}(\frac{4}{3})}^{\tan^{-1}(\frac{12}{3})}$$

(This one doesn't work out as nicely as I thought, but this answer is fine, even though the hint wasn't useful.)