Elementary Functions and Calculus I
Math 131 (Sec 42), Autumn 2004
Practice Mid-term 2 Solutions

1. Recall the definitions of the following.

   (a) That a function $f : \mathbf{R} \to \mathbf{R}$ has a limit l at a point x.
   Answer. We say $\lim_{y\to x} f(y) = l$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|y - x| < \delta$ then $|f(y) - f(x)| < \varepsilon$.

   (b) That a function $f : \mathbf{R} \to \mathbf{R}$ be continuous at a point x.
   Answer. If $\lim_{y\to x} f(y)$ exists and equals $f(x)$ then f is continuous at x.

   (c) That a function $f : \mathbf{R} \to \mathbf{R}$ be differentiable at a point x.
   Answer. The function f is differentiable at x if
   $$f'(x) := \lim_{y\to x} \frac{f(y) - f(x)}{y - x}$$
   exists and is not equal to $\pm\infty$.

2. Recall the following results (without proving them). State clearly any assumptions that are required for each result to hold.

   (a) The intermediate value theorem.
   Answer. If $f : I \to \mathbf{R}$, where I is an interval in $\mathbf{R}$, is continuous, $a, b \in I$ and M is between $f(a)$ and $f(b)$, then there exists a c such that $f(c) = M$.

   (b) The product rule.
   Answer. If f and g are functions, differentiable at c then fg is differentiable at c and
   $$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

   (c) The quotient rule.
   Answer. If f and g are functions differentiable at c and $g(c) \neq 0$, then f/g is differentiable at c and
   $$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{g^2(c)}.$$

(d) The chain rule.
Answer. If g is differentiable at c and f is differentiable at $g(c)$ then $f \circ g$ is differentiable at c and $(f \circ g)(c) = f'(g(c))g'(c)$

(e) The power rule.
Answer. Let r be a rational number and define $h : \mathbf{R} \to \mathbf{R}$ by $h(x) = x^r$ for all $x \in \mathbf{R}$. Then write $r = p/q$ where p and q have no common factors. If q is odd then h is differentiable at every $x \in \mathbf{R}$. If q is even then h is differentiable at every $x > 0$. Whenever h is differentiable $h'(x) = rx^{r-1}$.

3. Apply the intermediate value theorem to show that $p(x) = x^7 + 10x^3 + 2$ has at least one root.

Answer. Observe $p(-2) = -128 - 80 + 2 = -206$ and $p(1) = 13$. Also we know that polynomials are continuous. Therefore by the intermediate value theorem (applied with $M = 0$) there exists a c such that $p(c) = 0$.

4. Use any of the definitions or theorems you have quoted above to find the derivative f' of f when defined by the following formulae. State clearly which results you are using, and prove any steps which are not direct applications of the theorems.

(a) $f(x) = 10x^{25} + 50x^2 - 4x + 20x^3$
Answer. By the power rule and linearity of the derivative,

$$f'(x) = 250x^24 + 100x - 4 + 60x^2.$$

(b) $f(x) = (x^3 + 40x)^{308}$
Answer. Let $g(x) = x^3 + 40x$ and $h(x) = x^{308}$, then by the power rule and linearity $g'(x) = 3x^2 + 40$ and $h'(x) = 308x^{307}$. Thus

$$f'(x) = (h \circ g)'(x) \overset{\text{chain rule}}{=} h'(g(x))g'(x) = 308(x^3 + 40x)^{307}(3x^3 + 40).$$

(c) $f(x) = \frac{x^9+50}{x^2+4}$
Answer. Let $g(x) = x^9 + 50$ and $h(x) = x^2 + 4$, then by the power rule $g'(x) = 9x^8$ and $h'(x) = 2x$. Thus

$$f'(x) = \left(\frac{g}{h}\right)'(x) \overset{\text{quotient rule}}{=} \frac{h(x)g'(x) - h'(x)g(x)}{h^2(x)} = \frac{9x^8(x^2 + 4) - 2x(x^9 + 50)}{(x^2 + 4)^2}$$

5. Given that $y : \mathbf{R} \to \mathbf{R}$ is differentiable and satisfies $x^2 y(x) = 1 + y^2(x)x$ find $y'(c)$ in terms of c and $y(c)$.

Answer. Differentiating both sides of the equality gives

$$2xy(x) + x^2 y'(x) = 2y(x)y'(x)x + y^2(x),$$

and so $y'(x) = (y^2(x) - 2xy(x))/(x^2 - 2y(x)x)$ provided the denominator is non-zero.

6. Let $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = x(x+1)$ if $x \leq 0$ and $g(x) = ax + b$ if $x > 0$ for some given $a, b \in \mathbf{R}$. Find $g'(y)$ for any $y \neq 0$. For what values of a and b is g continuous at zero? For what values of a and b is g differentiable at zero? What must $g'(0)$ be if it exists? Explain your answer.

Answer. If $x < 0$ then $g(x) = x(x+1)$ so by the product rule $g'(x) = (x+1) + x = 2x + 1$. If $x > 0$ then $g(x) = ax + b$ and so $g'(x) = a$. Observe $g(0) = 0$, $\lim_{x \searrow 0} h(x) = \lim_{x \searrow 0} ax + b = b$ and $\lim_{x \nearrow 0} h(x) = \lim_{x \nearrow 0} x(x+1) = 0$. Thus $\lim_{x \to 0} h(x) = h(0)$ if and only if the left-hand and right-hand limit exist and equal the value $h(0)$, that is, if and only if $b = 0$. That is f is continuous at 0 exactly when $b = 0$ and a can be any number.

Now for g to be differentiable at 0 the limit $\lim_{x \to 0}(g(x) - b)/(x - 0)$ must exist. That is, the left-hand and the right-hand limits must exist and be equal. We can compute the right-hand limit is a and the left-hand limit is

$$\lim_{x \nearrow 0} \frac{x(x+1) - b}{x} = \lim_{x \nearrow 0} \frac{x^2 + x - b}{x} = \lim_{x \nearrow 0} \left(x + 1 - \frac{b}{x} \right)$$

which exists if and only if $b = 0$ (why?). When $b = 0$ the left-hand limit equals 1. Thus for the derivative to exist we must have $a = 1$ and $b = 0$.