**Elementary Functions and Calculus I**
**Math 131 (Sec 42), Autumn 2004**
Final Examination Revision Sheet

In order to study for this exam I recommend you do the following. You should read through all the notes from class and understand everything that is written there. Make new notes reminding yourself of the definitions of the course and the theorems (but not necessarily the proofs). Then redo the mid-terms and practice mid-terms and do the practice final. Remember to look over related rates and the max/min section as we have not covered them in the midterms.

As an absolute minimum you should be able to do the following.

1. Recall the definitions of the following.
   - That a function $f : \mathbf{R} \to \mathbf{R}$ has a limit $l$ at a point $x$.
   - That a function $f : \mathbf{R} \to \mathbf{R}$ be continuous at a point $x$.
   - That a function $f : \mathbf{R} \to \mathbf{R}$ be differentiable at a point $x$.
   - An extreme value and a critical point.
2. Recall the following results (without proving them). State clearly any assumptions that are required for each result to hold.
   - The intermediate value theorem.
   - The product rule.
   - The quotient rule.
   - The chain rule.
   - The power rule.
   - The critical point theorem.

Practice Final Examination

1. Apply the intermediate value theorem to show that $p(x) = x^3 + 3$ has at least one root.

2. Use any of the definitions or theorems you have quoted above to find the derivative $f'$ of $f$ when defined by the following formulae. State clearly which results you are using, and prove any steps which are not direct applications of the theorems.

   (a) $f(x) = 3x^{27} + 32x^5 - 4x + 20x^3$

   (b) $f(x) = (x^2 + 6x)^{35}$

   (c) $f(x) = \frac{x^7+34}{x^2+5}$

3. Given that $y : \mathbf{R} \to \mathbf{R}$ is differentiable and satisfies $x^3 y(x) = 1 + y^2(x)x$ find $y'(c)$ in terms of $c$ and $y(c)$.

4. Let $g : \mathbf{R} \to \mathbf{R}$ be defined by $g(x) = x^2$ if $x \leq 0$ and $g(x) = ax + b$ if $x > 0$ for some given $a, b \in \mathbf{R}$. Find $g'(y)$ for any $y \neq 0$. For what values of $a$ and $b$ is $g$ continuous at zero? For what values of $a$ and $b$ is $g$ differentiable at zero? What must $g'(0)$ be if it exists? Explain your answer.

5. (a) Find all the values of $d \in \mathbf{R}$ such that $|d - 7| > 7$.

   (b) Find all the values of $x \in \mathbf{R}$ such that $x^2+8x+22 > x^2+9x+18 > 0$.

6. Use only the basic properties of the real numbers (commutativity, associativity, distributivity, identity elements and inverses) and of the order relation $<$ (trichotomy, transitivity, addition and multiplication) to prove that if $a < b$ and $c < d$, where $a$, $b$, $c$ and $d$ are positive numbers, then $ac < bd$.

7. (a) Let $f : \mathbf{R} \to \mathbf{R}$ be a function. Define what it means that $f$ has a limit $l$ at a point $x$. Which number in your definition is written as $\lim_{y \to x} f(y)$?

   (b) Prove directly from the definition you gave above that

$$\lim_{x \to -2} \frac{x^2 + 4x + 4}{x + 2} = 0.$$

8. (a) Complete the following definitions. (Hint: we are writing $\lim_{y \to x} f(y) = f(x)$ out in terms of the definition of the limit.)

i. We say that a function $f$ is continuous at $x$ if for any $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that…

ii. We say that a function $g$ is continuous at $x$ if for any $\varepsilon > 0$ there exists a $\delta_2 > 0$ such that…

(b) Now, assuming $f$ and $g$ are indeed continuous at $x$ prove directly from the $\varepsilon - \delta$ definition that this implies $f + g$ is continuous at $x$. (Hint: It is enough to prove $\lim_{y\to x}(f + g)(y) = f(x) + g(x)$. With notation as in (a), a good choice of $\delta$ is $\delta = \min\{\delta_1, \delta_2\}$.)

9. For this question you may use the main limit theorem without proof. Indicate clearly where you do use it and, if necessary, the required conditions. Compute the following limits:

   (a) $\lim_{x\to 6} \frac{(x-4)(x-10)}{(x^2+4)}$;

   (b) $\lim_{x\to 7} \frac{|x-7|(x-5)}{|x-5|}$;

   (c) $\lim_{x\to 10} \sqrt{x^2 + 22}$.

10. Suppose $a$ and $b$ are given numbers. Define $f : [-1, 1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} (1-x)x, & \text{if} \quad x \leq 0; \\ ax + b, & \text{if} \quad x > 0. \end{cases}$$

   (a) Sketch the graph of $f$. Label the value of $b$ on the $y$-axis.

   (b) Is $f$ continuous at $x$ if $x \neq 0$? Justify your answer.

   (c) For what values of $b$ is $f$ continuous at 0? Justify your answer.

11. (a) Define what it means for a function $f : \mathbf{R} \to \mathbf{R}$ to be differential at a point $x$. What is the derivative of $f$ at $x$?

    (b) Prove directly from the definition you just gave that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x$ for all $x \in \mathbf{R}$ is differentiable at every $c \in \mathbf{R}$.

12. The aim of this question is to prove that if a function $f$ is differentiable at a point $x$ then $f$ is continuous at $x$.

(a) Let $x, y \in \mathbf{R}$ be such that $x \neq y$. Simplify the following expression.

$$f(x) + (y - x)\left(\frac{f(y) - f(x)}{y - x}\right)$$

(b) Assume now $f'(x)$ exists, that is $f$ is differentiable at $x$. Using the formula you have just computed in (a), show that $\lim_{y \to x} f(y)$ exists. (Hint: take the limit of both sides of the formula in part (a).)

(c) In showing the limit existed what is the value of $\lim_{y \to x} f(y)$ you computed above? Thus what have you proven?

13. (a) Assume $f$ and $g$ are functions differentiable at $c$. State the product rule.

    (b) Suppose $f(4) = 5$, $f'(4) = 3$, $g(4) = 7$ and $g'(4) = 40$.

       (i) Are $f$ and $g$ continuous at 4? Justify your answer.

       (ii) Calculate $(fg)'(x)$.

14. (a) State the power rule for *integers*. That is, let $h : (0, \infty) \to \mathbf{R}$ be given by $h(x) = x^n$ where $n$ is an integer. Give a formula for $h'(x)$. Is this formula valid on the given domain of $h$?

    (b) State the chain rule for two function $f$ and $g$, $f$ differentiable at $g(x)$ and $g$ differentiable at $x$.

    (c) Use these rules to calculate $(f \circ g)'(x)$ where $g(x) = x^3 + 5x$ and $f(x) = x^{400}$ for $x > 0$.

15. (a) Define what it means for $f : [a, b] \to \mathbf{R}$ to have a maximum value $M \in \mathbf{R}$.

    (b) Define what it means for $f : [a, b] \to \mathbf{R}$ to have a critical point $c \in [a, b]$

    (c) State the critical point theorem.

    (d) Prove the maximum of $g : [-2, 2] \to \mathbf{R}$ given by $g(x) = x^2$ is 4. (Be careful to explain why you know the maximum exists).

    (e) With $g$ defined as above, on which critical points does $g$ attain it maximum?